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A joint analysis of the S–wave in the $\pi^+\pi^-$ and $\pi^0\pi^0$ data

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Abstract

We use our former results on $\pi^+\pi^-$ S–wave obtained in a nearly assumption-free way from the 17.2 GeV/c data to predict the $\pi^0\pi^0$ S–wave. The predictions are compared with the recent results of the E852 experiment at 18.3 GeV/c. A good agreement is found for only one (the "down–flat") solution while the second one (the "up–flat") is excluded by the $\pi^0\pi^0$ data. Thus the long-standing "up-down" ambiguity has been finally resolved in favour of the S–wave intensity which stays large and nearly constant up to the $K\overline{K}$ threshold. A joint analysis of both sets of data leads to a reduction of errors for this solution.

1 Introduction

Despite of several decades of efforts a spectrum and properties of scalar mesons are still not well known. This is particularly true in the low mass region (below the $K\overline{K}$ threshold), where the whole information comes from the $\pi\pi$ partial wave analysis (PWA). It is interesting to note that the best experimental data on the $\pi^+\pi^-$ system were collected in the seventies. We mean here e.g. the CERN-Munich experiment [1], which supplied 3×10^5 events of the reaction

$$\pi^- p \to \pi^+\pi^- n \tag{1}$$

at 17.2 GeV/c. It should be stressed that the number of observables provided by such experiment is much smaller than the number of real parameters needed to describe the partial waves. Consequently, special physical assumptions[1] were made in most studies [1-4] of the $\pi^+\pi^-$ system and are still made in all PWA's

[1]Hereafter called the *standard* assumptions.

of the $\pi^0\pi^0$ system. It has been customary to produce results *without even mentioning the assumptions* essential for their derivation.

These assumptions are generally connected with ignoring of the role of nucleon spin. Also the a_1 exchange amplitude (which is dominantly spin-nonflip) was neglected thus assuming the full dominance of the pion exchange. It should be stressed that the reality can be more complicated, the nucleon spin cannot be *a priori* ignored and there is a significant pseudovector (the a_1) exchange in addition to the dominant pseudoscalar pion exchange. This was demonstrated by another important experiment, namely that of the CERN-Cracow-Munich (CCM) collaboration [5] which yielded 1.2×10^6 events of the reaction:

$$\pi^- p_\uparrow \to \pi^+\pi^- n \tag{2}$$

also at 17.2 GeV/c. The presence of the transversely polarized target provides additional observables. It was shown that there is a considerable a_1 exchange. Combination of the results of both experiments enables a quasi-complete PWA with much weaker physical assumptions. The analysis is only a quasi-complete one because we do not know the phase between two sets of transversity amplitudes. Nevertheless e.g. the *full* (containing both pseudoscalar and pseudovector exchange as well as both $I = 0$ and $I = 2$ components for S–wave and D–waves) partial wave intensities could be determined in a completely model-independent way. This was done in another CCM paper [6]. While for dipion mass $m_{\pi\pi} = (1000 \div 1800)$ MeV a unique solution was obtained, the up-down ambiguity reappeared below 1 GeV.

In our first paper [7] (hereafter called paper I) we "have constructed a bridge" between two sets of transversity amplitudes for the $S-$wave. This was done by first demanding the phases of the leading P–, D– and F–transversity amplitudes to follow exactly the phases of the Breit–Wigner $\rho(770)$, $f_2(1270)$ and $\rho_3(1690)$ resonant amplitudes in the low, medium and high mass region, respectively. Then, from the measured phase differences between these waves and the S–wave we calculated the *absolute* phases of the S–wave transversity amplitudes[2]. Finally the pseudoscalar and pseudovector exchange amplitudes in the S–wave were explicitly determined from the relevant combinations of the S–wave transversity amplitudes. In order to calculate the $I = 0$ components we have used the $I = 2$ S–wave as measured by the CERN-Munich [8] collaboration. This separation, first ever made, has been done using much weaker assumptions[3] than those made in *any* earlier analysis.

[2]Let us note that such a procedure would be impossible for the low mass $\pi^0\pi^0$ S–wave even if the reaction (3) was measured on a polarized target. There would be no $P-$wave to be used as a handle to determine the absolute phase of the S–wave .

[3]The main assumption was neglecting of a possible influence of the a_1 exchange in $I = 2$ S–wave . This assumption was also made in all other studies.

The price we paid was a fourfold ambiguity in our pseudoscalar exchange S–wave amplitude. In addition to the "up-down" ambiguity[4] of the old CCM analysis [6] there were ambiguities resulting from adding or subtracting the phase difference since the PWA of the CCM group yielded only the absolute value of the phase difference. Thus we have "down–flat" , "down–steep" , "up–flat" and "up–steep" solutions. Differences between "flat" and "steep" refer mainly to the behaviour of the S–wave phase shifts below the $f_0(980)$. Above the $f_0(980)$ resonance all the solutions are fairly similar. The main difference is that both "steep" solutions contain a relatively narrow[5] resonance under the $\rho(770)$ (like the old "up" solution). Such resonance was persistently claimed by Svec [9]. On the other hand, both "flat" solutions indicate a $f_0(500)$ state [10] with a width of about 500 MeV. In particular the "down-flat" solution is essentially similar to the old solution of the CERN-Munich group [1] obtained with the help of the standard assumptions.

Having arrived at as many as four solutions we have immediately started a thorough investigation of them in a hope of reducing their number. Already in paper I we have shown the "down–steep" solution to violate unitarity while both "flat" solutions easily passed this test. The "up–steep" solution although behaving in a queer way could not be immediately excluded. Our next paper [11] (hereafter called paper II) was devoted to a detailed study of this suspicious solution. Using the experimental fact that there is no significant inelastic channel below the $f_0(980)$ we have imposed the $\eta = 1$ inelasticity for all mass points in this mass region. For the "up–steep" solution this led to unphysical results in 7 out of 20 mass bins and queer behaviour in the remaining mass bins. The effect was even stronger for the "down–steep" solution, already rejected in paper I. Thus removing both "steep" solutions we were left with two "flat" solutions equally well describing the $\pi^+\pi^-$ data. The further step demanded an extra input.

Fortunately such an input has been recently provided by the BNL E852 collaboration [12], which has collected 8.5×10^5 events of the reaction

$$\pi^- p \to \pi^0 \pi^0 n \tag{3}$$

at 18.3 GeV/c. A comparison of both "flat" solutions to the $\pi^0\pi^0$ data is a topic of the present paper.

The paper is organized as follows. In Sect. 2 we describe the $\pi^+\pi^-$ and $\pi^0\pi^0$ data. In Sect. 3 we relate our $\pi^+\pi^-$ S–wave amplitudes to the amplitudes describing the $\pi^0\pi^0$ channel, in which the role of the isospin 2 component is substantially enhanced in comparison with the $\pi^+\pi^-$ channel. In Sect. 4 we

[4]The "down" S–wave intensity stays high and nearly constant from $m_{\pi\pi} \approx 750$ MeV to the $f_0(980)$ while the "up" solution exhibits a resonant-like maximum under the $\rho(770)$.

[5]Hereafter "relatively narrow" means "with a width close to $\Gamma_\rho = 150$ MeV".

compare our predictions to the $\pi^0\pi^0$ data and use both sets of data to improve our knowledge of the $\pi\pi$ S–wave . The results are summarized in Sect. 5.

2 Description of the $\pi\pi$ data, first comparison and relative normalization

There were several experiments yielding $\pi^0\pi^0$ data from reaction (3). The highest statistics was provided by the GAMS experiment which collected 1.5×10^6 events [13] at 38 GeV/c and 6.4×10^5 events [14] at 100 GeV/c. However these high-energy experiments were mainly dealing with the high $m_{\pi\pi}$ region. The NICE experiment [15] at 25 GeV/c, the KEK E135 experiment [16] and some, even older, experiments were of inferior statistics. In this paper we use the results of the E852 collaboration [12] obtained at 18.3 GeV/c i.e. very close to the CCM momentum of 17.2 GeV/c. Additional (and essential) advantage of these results is their availability on the World Wide Web where explicit numbers are given. These are intensities of partial waves in $\Delta m_{\pi\pi} = 40$ MeV bins for four intervals of the four-momentum transfer t between –0.01 GeV/c^2 and –1.5 GeV/c^2.

For a comparison with our results we have merged two $|t|$ bins i.e. $(0.01 \div 0.10)$ GeV^2/c^2 and $(0.10 \div 0.20)$ GeV^2/c^2 into a single one i.e. $(0.01 \div 0.20)$ GeV^2/c^2. This was done by adding the values and their errors linearly and in quadrature, respectively. This binning is exactly equal to that in the original CCM paper [6] while in papers I and II we used $\Delta m_{\pi\pi} = 20$ MeV bins and $|t| = (0.005 \div 0.200)$GeV^2/c^2.

It should be noted that both experiments have certain problems at low $m_{\pi\pi}$. The spark chamber CM experiment [1] had low and badly controlled acceptance for two close tracks; thus it was decided not to publish data below 600 MeV. The E852 data [12] start right from the threshold but they are somewhat disturbed by some $K^0_s \to \pi^0\pi^0$ decays which escaped rejection. The upper $m_{\pi\pi}$ limit of 1800 MeV (1600 MeV) for $|t| > 0.01$ GeV^2/c^2 ($|t| > 0.005$ GeV^2/c^2) in the CCM data is of lesser importance for our study of ambiguities present mainly below the $f_0(980)$. However a warning should be added concerning the $\pi^0\pi^0$ data at $m_{\pi\pi} = (1400 \div 1600)$ MeV. An inspection of Fig. 5D and Fig. 7D of ref. [12] shows a substantial G $(J = 4)$ wave below 1600 MeV, which seems to be very unlikely at such a low mass. Consequently the $\pi^0\pi^0$ S–wave is probably underestimated in this mass region.

Unfortunately the E852 collaboration has provided only the acceptance-corrected numbers of events and *not the absolute cross section* which is available for the CCM data. Thus we have determined the normalization to our data using the strongest D_0 wave, generally well fixed by the t^4_0 moment and saturated by the $f_2(1270)$. The fit in the range $m_{\pi\pi} = (1000 \div 1600)$ MeV yields the normalization factor $F_d = (8.58 \pm 0.16) \times 10^{-6} \mu b$/event in 20 MeV bins. This normalization factor was then applied to other waves. The results are shown in Fig. 1.

An inspection of Fig.1 shows:

– a good agreement *in the shape* of the D_0 wave; this shows that this wave can be used for normalization,

– a systematic difference in the D_U wave (unnatural spin parity exchange) with virtually all E852 points below the $\pi^+\pi^-$ data,

– a similar but much less significant effect in the D_N wave (natural spin parity exchange).

Now let us concentrate on the D_U wave. We attribute the difference to the assumptions described in the introduction, which were used in the PWA for $\pi^0\pi^0$ but not for the $\pi^+\pi^-$ data. This effect was already observed in the CCM paper (see Fig. 5 of ref. [6]), when the intensity of the D_U wave obtained using the data from the polarized target experiment was compared to that from the analysis [3] based on the standard assumptions. The assumptions led to an underestimation of the D_U wave and the overestimation of the least-fixed $S-$ wave, both the total intensity and that of the D_0- wave being fixed by the relevant moments. Assuming the same mechanism we have tentatively calculated the excess of the $\pi^+\pi^-$ D_U intensity over the $\pi^0\pi^0$ one and subtracted it from the S–wave intensity in the latter. This leads to a $(17 \pm 4)\%$ reduction of the S–wave in the $\pi^0\pi^0$ plot and the normalization factor $F_s^{'} = (7.13 \pm 0.35) \times 10^{-6}$ μb/20 MeV per one event. Including the above underestimation of the D_N wave by about 10 % yields $F_s^{''} = (6.44 \pm 0.52) \times 10^{-6}$ μb/20 MeV per one event. Although this whole procedure looks like a step in the wrong direction (generally the $\pi^0\pi^0$ data are already *below* the $\pi^+\pi^-$ ones) we shall see later that a good agreement in one of the S–wave solutions can be reached after properly handling the isospin $I = 2$ component of the S–wave amplitudes. Since we do not know whether $F_s^{'}$ or $F_s^{''}$ is the proper value we use the intermediate value $F_s = (6.70 \pm 0.78) \times 10^{-6}$ μb/20 MeV. The error range covers both values including their error bars. The value F_s was used to normalize the E852 data and Fig.2 shows a comparison of the cross sections found in paper I for two solutions: "up–flat" and "down-flat". The errors of the $\pi^0\pi^0$ intensities were obtained by adding the normalization and statistical errors in quadrature. It should be noted that we have repeated the calculation of the normalization factor for various mass ranges as well as for the CCM binning ($\Delta m_{\pi\pi} = 40$ MeV and $|t| > 0.01$ GeV^2/c^2); the results change very little.

The correctness of the above normalization is supported by the following reasoning. Since the $\pi\pi$ production mechanism in the D–wave is very similar to that in the S–wave we can estimate the isospin 2 contribution to the D–wave amplitude in the $\pi^+\pi^-$ and $\pi^0\pi^0$ intensities dominated above 1 GeV by the isospin 0 resonance $f_2(1270)$. It was found in Ref. [8] that in the $\pi\pi$ effective

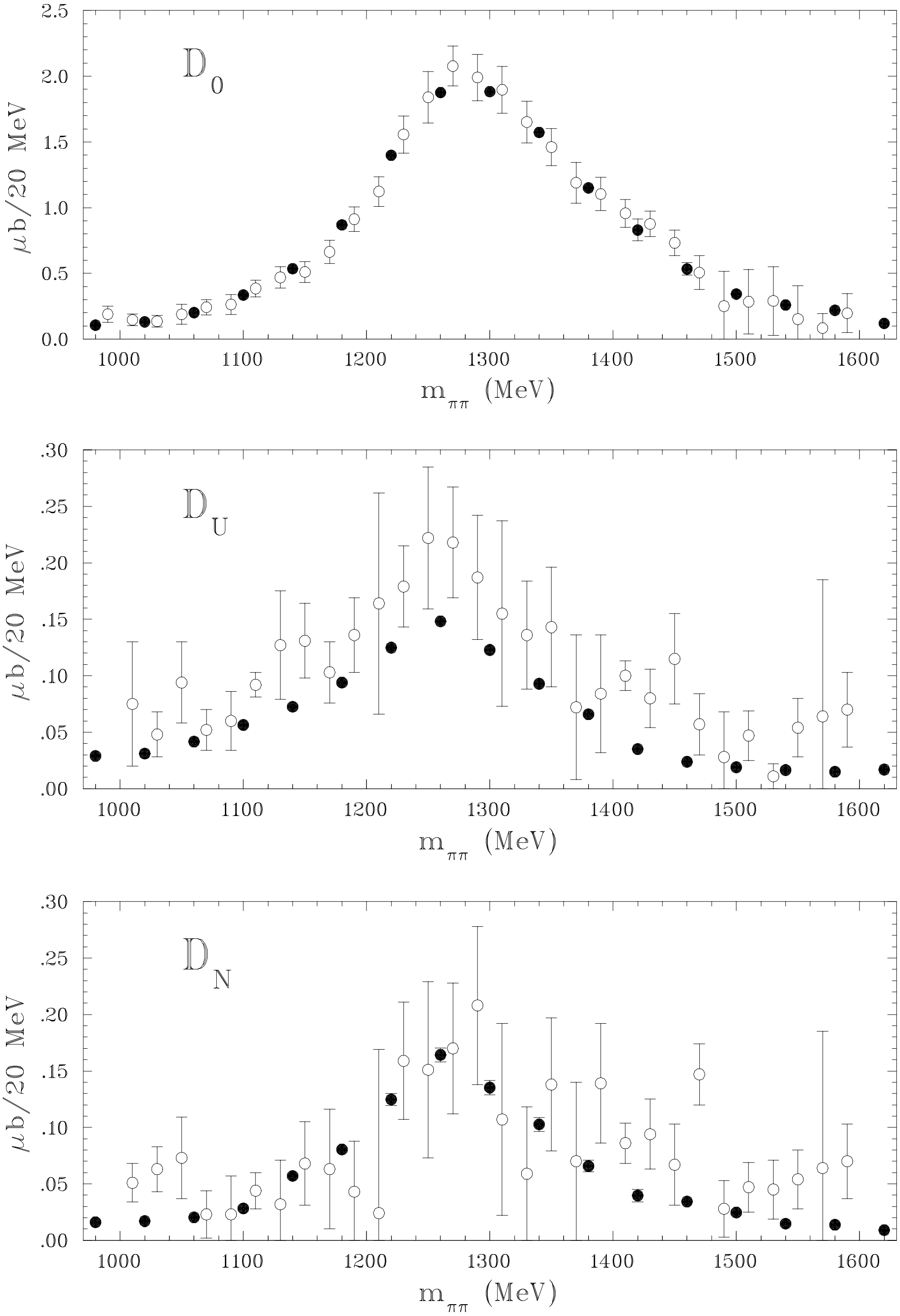


Fig. 1: Intensity of D–waves for the $\pi^0\pi^0$ data (full circles) and the $\pi^+\pi^-$ data (open circles)

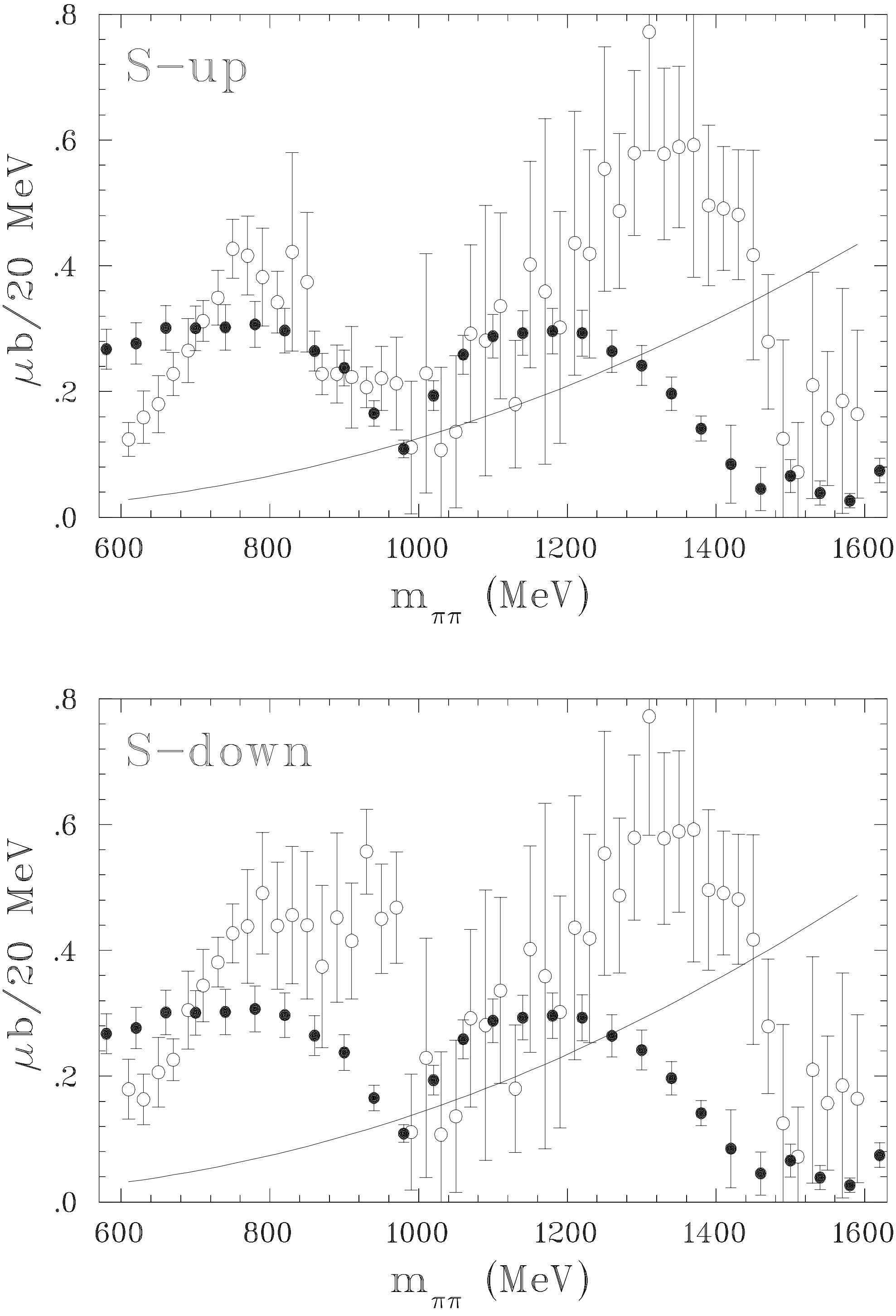


Fig. 2: Intensity of S–wave for the $\pi^0\pi^0$ data (full circles) and the $\pi^+\pi^-$ data (open circles). Solid line denotes the $I = 2$ contribution $2\times\ |0.5\overline{C_\pi U}T_2|^2$, see Eqs. (26), (27) and (30).

mass range between 1.25 and 1.50 GeV the $I=2$ D–wave phase shift is negative and its absolute value does not exceed 4 degrees. We have estimated the D–wave isospin 2 contribution assuming that the D–wave a_1 exchange amplitude is smaller than the corresponding one pion exchange amplitude and that the dependence of the vertex form factors on t is the same for the $\pi^+\pi^-$ and $\pi^0\pi^0$ production in both S– and D–waves. It turned out that the D_0 wave intensities in both reactions should be equal within about 2 %. The assumption about the vertex form factors is justified by a very similar shape (to about 5 % below $|t| = 0.4$ GeV^2) of the t- distributions in the $f_2(1270)$ and $\rho(770)$ mass ranges as shown in Ref. [1].

Let us also comment on the errors being much smaller for the E852 results. This is mainly due to high statistics and to smaller numbers of partial waves (e.g. the low mass S–wave saturates the $\pi^0\pi^0$ intensity while in the $\pi^+\pi^-$ data it competes with much stronger P–wave). On the other hand the E852 errors do not include the systematic errors of the standard assumptions inherent in their PWA. These assumptions were absent in the CCM analysis. In comparison of the both sets of data the large normalization errors are very important.

3 Calculation of S–wave intensity for $\pi^0\pi^0$ channel from our analysis of the $\pi^+\pi^-$ data

In this chapter we shall discuss relations between the $\pi\pi$ production amplitudes in various charge combinations. In addition to the reactions (1) and (3) we consider the following reaction:

$$\pi^+ p \to \pi^+\pi^+ n \tag{4}$$

which is an essential source of information about the $\pi\pi$ phase shifts in the scalar isospin $I=2$ $\pi\pi$ channel [8].

At high energies of incoming pions and at small momentum transfers to the target the above three reactions are dominated by the one pion exchange mechanism. However, as shown in [6] and in I the pseudovector $a_1(1260)$ exchange, although smaller than the one pion exchange, is also present and cannot be neglected. Let us denote by $T(\pi\pi \to \pi\pi)$ the off-shell elastic or charge exchange $\pi\pi$ amplitudes and by $t(a_1\pi \to \pi\pi)$ similar amplitudes in which the exchanged a_1 is off-shell. The coupling of the exchanged pion to the target proton and neutron is relatively well known. This is not the case for the a_1 coupling but in paper I we have been able to separate the a_1 exchange amplitudes using the polarized target data of reaction (2). Now we shall use those amplitudes and assuming the isospin symmetries of the T and t amplitudes we shall write down the amplitudes corresponding to the $\pi^0\pi^0$ production. Using the isospin symmetry [17] we can write:

$$T(\pi^+\pi^- \to \pi^+\pi^-) = \frac{1}{3}T_0 + \frac{1}{2}T_1 + \frac{1}{6}T_2, \tag{5}$$

$$T(\pi^+\pi^- \to \pi^0\pi^0) = -\frac{1}{3}T_0 + \frac{1}{3}T_2, \tag{6}$$

$$T(\pi^+\pi^+ \to \pi^+\pi^+) = T_2, \tag{7}$$

where T_j $(j = 0,1,2)$ are the $\pi\pi$ isospin $(I = 0,1,2)$ amplitudes. They are normalized to Argand's form:

$$T_j = \frac{\eta_j e^{2i\delta_j} - 1}{2\ i}, \tag{8}$$

where η_j are inelasticities and δ_j are the channel j phase shifts. Similarly one can write the $a_1\pi \to \pi\pi$ amplitudes as follows

$$t({a_1}^+\pi^- \to \pi^+\pi^-) = \frac{1}{3}t_0 + \frac{1}{2}t_1 + \frac{1}{6}t_2, \tag{9}$$

$$t({a_1}^+\pi^- \to \pi^0\pi^0) = -\frac{1}{3}t_0 + \frac{1}{3}t_2 \tag{10}$$

$$t({a_1}^+\pi^+ \to \pi^+\pi^+) = t_2, \tag{11}$$

where t_j $(j = 0,1,2)$ are again the isospin amplitudes corresponding to the production of the $\pi\pi$ pair by the a_1 meson scattered on a pion. Let us notice that the isotensor amplitudes T_2 and t_2 enter in different combinations in the above equations. After the partial wave projection the isospin $I = 1$ amplitudes will not contribute to the S–wave $\pi\pi$ production so the T_1 and t_1 amplitudes can be ignored.

As in paper I we consider two independent S–wave $\pi\pi$ transversity amplitudes g and h for the reaction (2) which differ by a proton or neutron spin projection on the axis perpendicular to the production plane ($g \equiv< n \downarrow |T|p \uparrow>$ and $h \equiv< n \uparrow |T|p \downarrow>$). By g_0 and h_0 we denote the amplitudes corresponding to the reaction (3) and by g_2 and h_2 the amplitudes of the reaction (4).

The amplitudes of the type g or h for those three reactions can be expressed as sums of two terms proportional to the proper combinations of the amplitudes T_i and t_i. Thus according to paper I :

$$g = C_\pi U(\frac{1}{3}T_0 + \frac{1}{6}T_2) + C_a V(\frac{1}{3}t_0 + \frac{1}{6}t_2), \tag{12}$$

$$h = C_\pi U^*(\frac{1}{3}T_0 + \frac{1}{6}T_2) + C_a V^*(\frac{1}{3}t_0 + \frac{1}{6}t_2), \tag{13}$$

where the functions C_π and C_a are proportional to the propagators of the exchanged π and a_1 , respectively. The expression for the C_π factor is following:

$$C_\pi = -\frac{m_{\pi\pi}\sqrt{2 \cdot \frac{g^2}{4\pi}}}{p_\pi \sqrt{s q_\pi}\, f} \frac{e^{at}}{m_\pi^2 - t}, \tag{14}$$

where:
p_π is the incoming π^- momentum in the $\pi^- p$ c.m. frame,
m_π is the pion mass,
s is the square of the total energy in the $\pi^- p$ c.m. frame,
q_π is the final pion momentum in the $\pi\pi$ c.m. frame,
$g^2/4\pi = 14.6$ is the pion-nucleon coupling constant,[6]
and f is the complex correction factor. This phenomenological parameter should account for the averaged $t-$dependence of the pion-nucleon vertex function and an additional small phase of the pion propagator. In paper I this parameter, close to 1, was introduced to satisfy the requirement that the average inelasticity coefficient for the $S-$ wave isoscalar $\pi\pi$ amplitude should be equal to 1 below the $K\overline{K}$ threshold. As in paper I we take the slope parameter $a = 3.5$ GeV^{-2}. The complex functions U and V which depend on the kinematical variables like $m_{\pi\pi}$ and t are defined by Eqs. (16) and (17) of paper I. The exact functional form of C_a (which can be also obtained from paper I) will not be needed in further considerations.

Using Eqs. (6) and (10) we can express the amplitudes g_0 and h_0 for the reaction (3) in the form analogous to Eqs. (12) and (13):

$$g_0 = C_\pi U(-\frac{1}{3}T_0 + \frac{1}{3}T_2) + C_a V(-\frac{1}{3}t_0 + \frac{1}{3}t_2), \tag{15}$$

$$h_0 = C_\pi U^*(-\frac{1}{3}T_0 + \frac{1}{3}T_2) + C_a V^*(-\frac{1}{3}t_0 + \frac{1}{3}t_2). \tag{16}$$

Finally we give equations for the amplitudes of the reaction (4):

$$g_2 = C_\pi U T_2 + C_a V t_2, \tag{17}$$

$$h_2 = C_\pi U^* T_2 + C_a V t_2. \tag{18}$$

[6]This value of the pion-nucleon coupling constant has been used by us in paper I. According to new result of Ref. [18] $g^2/4\pi = 14.17 \pm 0.05 \pm 0.19$. Since C_π depends on the ratio $\sqrt{(g^2/4\pi)}/f$ any change in the value of the pion-nucleon coupling constant can be easily incorporated by a proper change of the modulus of f.

The S–wave double differential cross section for the reaction (3) reads:

$$\frac{d^2\sigma_0}{dm_{\pi\pi}dt} = |g_0|^2 + |h_0|^2. \tag{19}$$

In order to calculate it from (15) and (16) one needs to know four complex amplitudes T_0, T_2, t_0 and t_2. Even if one exploits the knowledge of the amplitudes g and h from our analysis of the polarized target data for the reaction (3) it is impossible to calculate (19) and find direct relations between g and g_0 or between h and h_0. This would be simpler if the E852 had made an experiment on polarized target. Since this was not a case we have to make an additional assumption to simplify the amplitudes. We shall neglect the amplitude t_2. This assumption was made in past in the approximate determination of the isotensor–scalar amplitude T_2 [8]. Now we do the same believing that t_2 is much less important that the amplitude t_0 which we do not neglect.[7]

Thus the equations (12), (13), (15) and (16) can be written as follows:

$$g = C_\pi U(\frac{1}{3}T_0 + \frac{1}{6}T_2) + C_a V(\frac{1}{3}t_0), \tag{20}$$

$$h = C_\pi U^*(\frac{1}{3}T_0 + \frac{1}{6}T_2) + C_a V^*(\frac{1}{3}t_0), \tag{21}$$

$$g_0 = C_\pi U(-\frac{1}{3}T_0 + \frac{1}{3}T_2) + C_a V(-\frac{1}{3}t_0), \tag{22}$$

$$h_0 = C_\pi U^*(-\frac{1}{3}T_0 + \frac{1}{3}T_2) + C_a V^*(-\frac{1}{3}t_0). \tag{23}$$

Now it is easy to relate amplitudes for two reactions in question:

$$g_0 = \frac{1}{2}C_\pi U T_2 - g, \tag{24}$$

$$h_0 = \frac{1}{2}C_\pi U^* T_2 - h. \tag{25}$$

From the above equations we infer that apart of the sign the amplitudes g, h and g_0, h_0, corresponding to the $\pi^+\pi^-$ and $\pi^0\pi^0$ production differ mainly by the presence of the term which includes the $I = 2$ amplitude and enters in different proportions in (5) and (6). This fact has an important consequence on the $m_{\pi\pi}$ dependence of the cross section (19).

In practical calculations we need amplitude averages over some ranges of the momentum transfer squared t. In paper I we have calculated the amplitudes $\overline{g}$ and $\overline{h}$ averaged over $-t$ between 0.005 and 0.2 GeV^2/c^2, so over $\Delta t_1 = (0.2 - 0.005)$ GeV^2/c^2= 0.195 GeV^2/c^2. Let us suppose that we need

[7] The standard approach was to ignore both amplitudes t_0 and t_2. In this analysis we ignore only t_2 which is a correction to t_0.

the amplitudes $\overline{g}_0$ and $\overline{h_0}$ averaged over some other range of $-t$ namely Δt (for example we shall take Δt=(0.2-0.01) GeV^2/c^2= 0.19 GeV^2/c^2). Then one obvious renormalization factor of the amplitudes is $n = (\Delta t/\Delta t_1)^{1/2}$. This factor will be used for the parts of g and h corresponding to the a_1 exchange where we do not expect a rapid variation with t (for details see paper I). The part corresponding to the one pion exchange (proportional to the function C_π) is more sensitive to the t-variation so we introduce another factor R equal to the ratio of the averaged product of $C_\pi U$ over the range Δt to the averaged product of $C_\pi U$ over the range Δt_1. Then using the definitions of the complex factors c_1 and c_2 (Eqs. (32) and (33) of paper I) one can derive the following relations:

$$\overline{g_0} = \frac{1}{2}\,\overline{C_\pi U}\;T_2 + b_g\;\overline{g} + b_h\;\overline{h} \tag{26}$$

and

$$\overline{h_0} = \frac{1}{2}\,\overline{C_\pi U^*}\;T_2 + {b_h}^*\;\overline{g} + {b_g}^*\;\overline{h}, \tag{27}$$

where

$$b_g = -(c_1 R + {c_1}^*)n, \tag{28}$$

$$b_h = c_2(1-R)n \tag{29}$$

and $\overline{C_\pi U}$ and $\overline{C_\pi U}^*$ denote the averages of the products $C_\pi U$ and $C_\pi U^*$ over the Δt range. Using (26) and (27) we obtain the effective mass distribution integrated over the momentum transfer squared:

$$\frac{d\sigma_0}{dm_{\pi\pi}} \equiv I_0 = |\overline{g_0}|^2 + |\overline{h_0}|^2. \tag{30}$$

We shall also call it the intensity I_0 of the S–wave in the reaction (3).

Thus using the analysis of the CCM $\pi^+\pi^-$ data performed in paper I and knowing the $I = 2$ component we can calculate the I_0 intensity for the reaction (3) using the phases of the $\overline{g}$ and $\overline{h}$ amplitudes fixed in paper I. Appearance of the new data on the $\pi^0\pi^0$ production enables us to relax some assumptions about those phases.

In paper I the phases of the transversity amplitudes $\overline{g}$ and $\overline{h}$ have been obtained essentially from the assumed resonant $\rho(770)$, $f_2(1270)$ and $\rho_3(1690)$ phases and the measured differences of the S–wave phase and the above mentioned P, D and F waves. However, in order to account for the P and the D wave phase differences in the range of $m_{\pi\pi}$ above 980 MeV, we have introduced an empirical additional phase Δ which was linearly decreasing function of $m_{\pi\pi}$ reaching the zero value at 1420 MeV (see Eq. (52) of I). Below 980 MeV where there was not enough available partial waves the value of Δ was arbitrarily assumed to be equal to 50.37°. Now we can directly calculate the values of the Δ function independently for each $m_{\pi\pi}$ bin using the new $\pi^0\pi^0$ results for I_0 in addition to the $\pi^+\pi^-$ data. In this way one can adjust the phases of $\overline{g}$ and $\overline{h}$ to fit *simultaneously* the $\pi^+\pi^-$ and the $\pi^0\pi^0$ cross sections. Let us note, however, that using the $\pi^0\pi^0$ intensity as a single input at a given effective $\pi\pi$ mass we cannot calculate two independent phases of $\overline{g}$ and $\overline{h}$. Therefore below 1460 MeV we follow the parametrization of paper I for the phases ϑ_g and ϑ_h of the amplitudes $\overline{g}$ and $\overline{h}$:

$$\vartheta_g = \begin{cases} \vartheta_g^S - \vartheta_g^P + \theta_{\rho(770)} & \text{for} \quad 600 \text{ MeV} \leq m_{\pi\pi} \leq 980 \text{ MeV}, \\ \vartheta_g^S - \vartheta_g^D + \theta_{f_2(1270)} + \Delta & \text{for} \quad 980 \text{ MeV} \leq m_{\pi\pi} \leq 1460 \text{ MeV}, \end{cases} \tag{31}$$

$$\vartheta_h = \begin{cases} \vartheta_h^S - \vartheta_h^P + \theta_{\rho(770)} + \Delta & \text{for} \quad 600 \text{ MeV} \leq m_{\pi\pi} \leq 980 \text{ MeV}, \\ \vartheta_h^S - \vartheta_h^D + \theta_{f_2(1270)} + \Delta & \text{for} \quad 980 \text{ MeV} \leq m_{\pi\pi} \leq 1460 \text{ MeV}. \end{cases} \tag{32}$$

In the above equations $\theta_{\rho(770)}$ and $\theta_{f_2(1270)}$ are phases of the resonant amplitudes defined in paper I and the remaining differences of the ϑ values represent the measured quantities in the partial wave analysis of the CCM group. The numerical values of Δ have been calculated to *reproduce* the experimental values of I_0. In subsequent fits we assume that the errors of phases ϑ_g and ϑ_h are solely given by the experimental errors of the differences between phases of the S and P or D waves determined in the partial wave analysis of the $\pi^+\pi^-$ data.

Linear interpolation of the 40 MeV bin data of the E852 group (including errors) has been applied in order to obtain the intermediate values corresponding to the 20 MeV $m_{\pi\pi}$ bins of the CERN-Cracow-Munich group. Following the arguments given in Section 2 the effective mass 1460 MeV is the maximum value at which we have used the E852 data.

4 A joint analysis of the $\pi^0\pi^0$ and $\pi^+\pi^-$ data

Fig. 3 shows the comparison of the $\pi^0\pi^0$ intensity I_0 from eq. (30) to the E852 data using the normalization factor F_s determined in Sect. 2. Comparing Fig. 2 and Fig. 3 we immediately see an essential role of the $I = 2$ component. Now it is the "down-flat" solution which reasonably agrees with the E852 data apart from the region above 1460 MeV, for which we suspect an underestimation of the S–wave in the E852 data (see Sect. 2). On the other hand the "up-flat" points are too low, especially in the region of $m_{\pi\pi} = (870 \div 970)$ MeV. We have fitted the E852 data to both $\pi^+\pi^-$ solutions with the normalization factor f_s as a free parameter. This has been done in two mass ranges:

- up to 980 MeV where the solutions are most different,
- up to 1460 MeV to avoid the uncertain mass region in the E852 data (see Sect. 2).

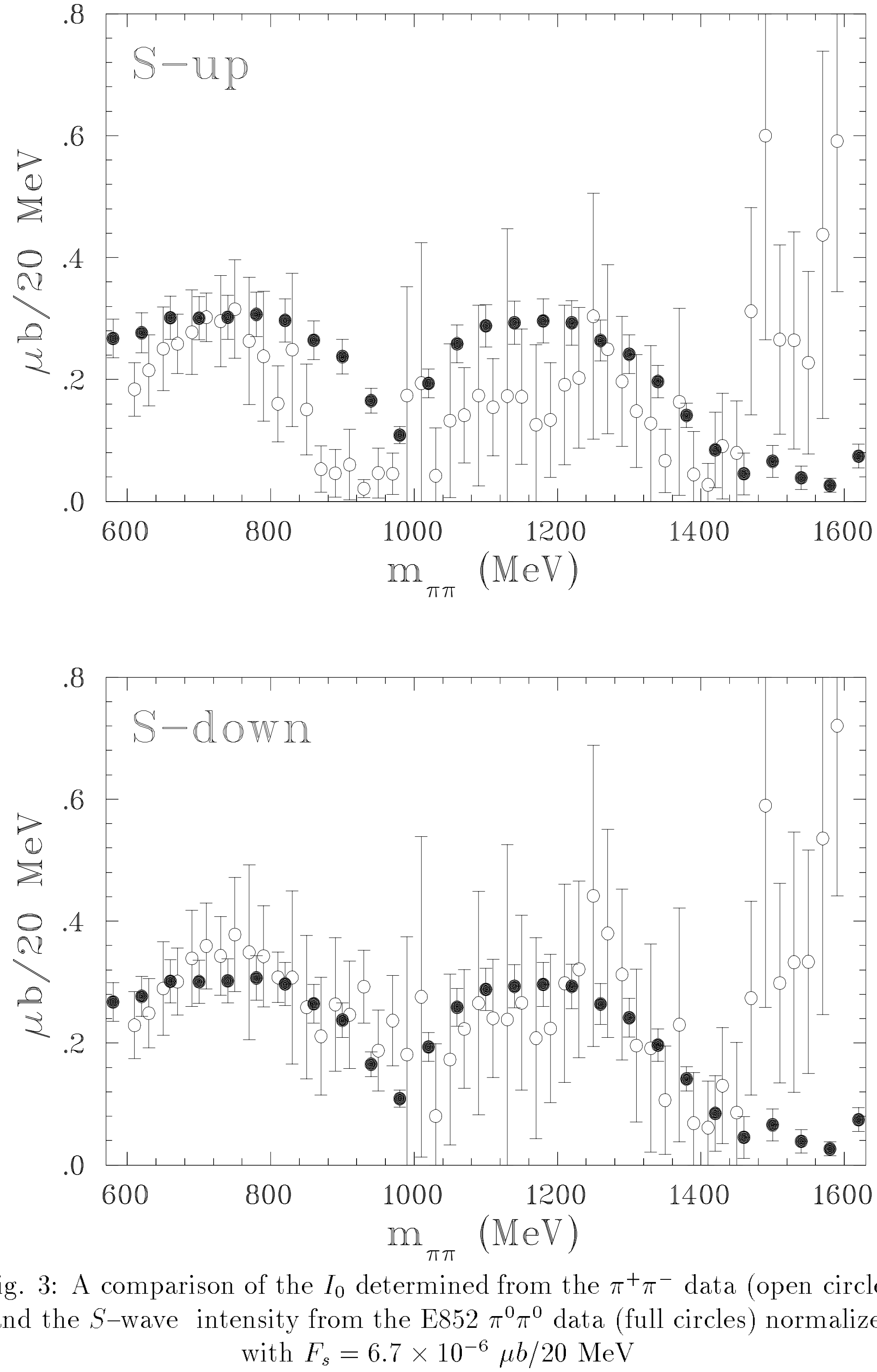


Fig. 3: A comparison of the I_0 determined from the $\pi^+\pi^-$ data (open circles) and the S–wave intensity from the E852 $\pi^0\pi^0$ data (full circles) normalized with $F_s = 6.7 \times 10^{-6}$ $\mu b/20$ MeV

Table 1

Table 1: Results of the fits to the $\pi^0\pi^0$ data

Solution	Mass range (MeV)	χ^2/NDF	f_s (units $10^{-6}\mu$b)
"down-flat"	$600 \div 980$	8.4/18	7.11 ± 0.42
"down-flat"	$600 \div 1460$	16.6/42	6.82 ± 0.37
"up-flat"	$600 \div 980$	39.2/18	4.33 ± 0.33
"up-flat"	$600 \div 1460$	45.9/42	4.16 ± 0.29

The results are shown in Table 1. While the fit is very good for the "down-flat" solution, the one for the "up-flat" version shows the strong disagreement. In the low mass range there is about 3σ deviation[8] with the normalization coefficient f_s being even more significantly (3.4σ) different from the $F_s^{''} = (6.44 \pm 0.52) \times 10^{-6}$ $\mu b/20$MeV which is the lowest possible normalization factor (see Sect. 2). In the whole mass range the fit is good but demands the normalization factor f_s which is 3.8σ below the corresponding $F_s^{''}$. If we had not corrected for the influence of the standard assumptions and used the F_d normalization factor then the $\pi^0\pi^0$ points would go up by approximately 20%. This could be still (although barely) reconciled with the "down–flat" solution but the deviation from the "up–flat" solution would be even more significant.

Let us stress that the fits to the "down–flat" solution yield the normalization factor f_s between $F_s = (7.13 \pm 0.35) \times 10^{-6}$ $\mu b/20$ MeV and $F_s^{''} = (6.44 \pm 0.52) \times 10^{-6}$ $\mu b/20$ MeV in good consistency with both. In particular the fit in the $(600 \div 1460)$ MeV mass range yields both value and error of f_s very close to $F_s = (6.70 \pm 0.78) \times 10^{-6}$ $\mu b/20$ MeV from Sect. 2.[9] This confirms that we have applied proper correction to the D_U and S–waves. Now all the D–waves (where the $I = 2$ component is negligible) nicely agree between two experiments while the S–wave for E852 data is entirely consistent with the I_0 for the "down-flat" solution of paper I.

Before concluding on a final rejection of the "up–flat" solution we have critically examined the assumptions introduced in paper I and repeated here. The aim of the exercise is an attempt to modify some of the assumptions in such a way that the "up–flat" solution is saved and we reproduce consistently both $\pi^0\pi^0$ and $\pi^+\pi^-$ $m_{\pi\pi}$ distributions. An obvious candidate is our assumption concerning the Δ (see eq. (31) and eq. (32)) - phase difference appearing in the S–wave g and h amplitudes.

[8] Hereafter σ denotes a standard deviation.

[9] This value of f_s means raising the E852 D_U wave exactly to the level of the $\pi^+\pi^-$ data and a small increase of the E852 D_N wave - see Fig. 1.

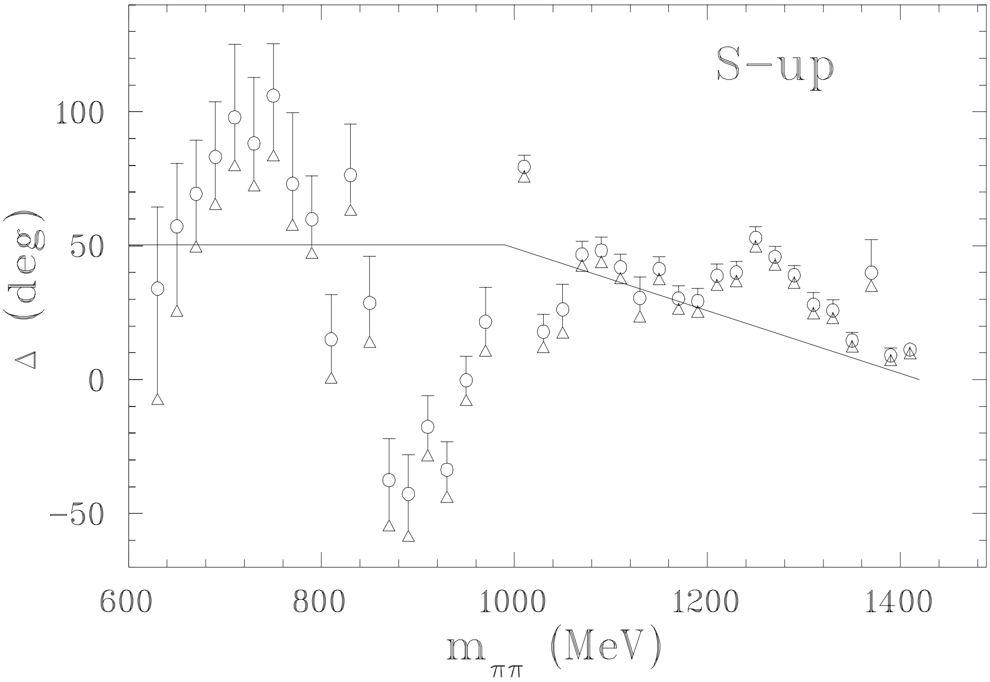
In paper I we dealt with only four observables from the CCM data in each mass bin. These are:
$I = |g|^2 + |h|^2$ - intensity of the $\pi^+\pi^-$ S–wave ,
$|g|/|h|$ - ratio of the moduli of transversity amplitudes of the above wave,
two phase differences (for g and h) between S–wave and the higher wave which is resonating in the region in question.

These four observables were used to determine four real parameters (two complex amplitudes) so we had to make some assumptions on Δ, which enters the expressions for the last two observables. Our ansatz assumed the Δ for the S–wave to follow the behaviour of the higher partial waves (P and D) for $m_{\pi\pi} > 980$ MeV. However for lower mass the D–wave is too weak to serve as a reference wave and we have arbitrarily assumed Δ to be constant and equal to that at $m_{\pi\pi} = 990$ MeV. As shown above it is in the low mass region that the "up–flat" solution disagrees with the E852 data. Therefore we check:
– whether one can find the values of Δ providing an agreement of the E852 data with the "up–flat" solution as well as with the "down-flat" solution,
– whether such set of values leads to reasonable behaviour of main parameters, namely the inelasticity η and the phase shift δ.

This was done by using the intensity I_0 of the $\pi^0\pi^0$ S–wave from the E852 data in addition to the observables listed above. A minor difference with respect to paper I is using the actual resonance parameters from the last Review of Particle Physics edition [19]. Also the range parameter for the $\rho(770)$ meson width was chosen equal to 5 GeV^{-1} for both solutions "down" and "up" (the Breit-Wigner parametrization of the resonant amplitudes is written in paper I).

We have critically reexamined all the solutions of partial wave analysis done by the CCM group. In case of more than one solution for a given ("up" or "down") branch we have consequently used the continuity criterion. Since these solutions had overlapping errors, the changes are small. At five points of $m_{\pi\pi}$ (at 630 MeV for the "down-flat" solution and at 990, 1130, 1250 and 1330 MeV for the solutions "down-flat" and "up-flat") new values of the CKM data were chosen. However, those differences in data have changed the values of δ and η by much less than their errors calculated in paper I.

We proceed in the following way: first the moduli of the transversity amplitudes g and h are calculated from the first two observables, then the $\pi^0\pi^0$ data are used to determine the values of Δ in each mass bin. We use the normalization factor $F_s = 6.70 \times 10^{-6}$ $\mu b/20$ MeV and calculate the vertex correction factor $f = |f|e^{i\phi}$ (see eq. 14) by minimizing the $(\eta-1)^2$ difference for 19 points of $m_{\pi\pi} < 980$ MeV. Now we have:
$|f| = 0.814 \pm 0.019$, $\phi = (-5.4 \pm 6.8)^o$ for the "down–flat" solution compared to $|f| = 0.84$, $\phi = -17.8^o$ in paper I,
$|f| = 0.924 \pm 0.027$, $\phi = (-32.3 \pm 4.3)^o$ for the "up–flat" solution compared to $|f| = 0.89$, $\phi = -4.4^o$ in paper I.



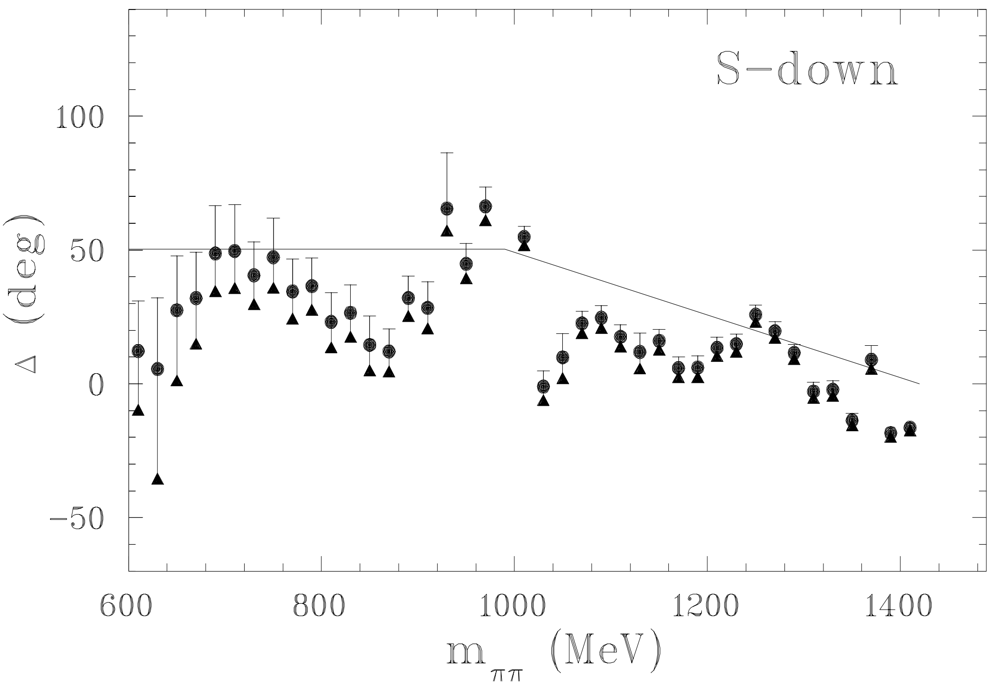


Fig. 4: A comparison of the parameter Δ (circles) with the empirical function of paper I (continuous line). Triangles correspond to the maximal values of the $\pi^0\pi^0$ intensity I_0 while the upper bars denote the values of Δ related to the lowest values of I_0.

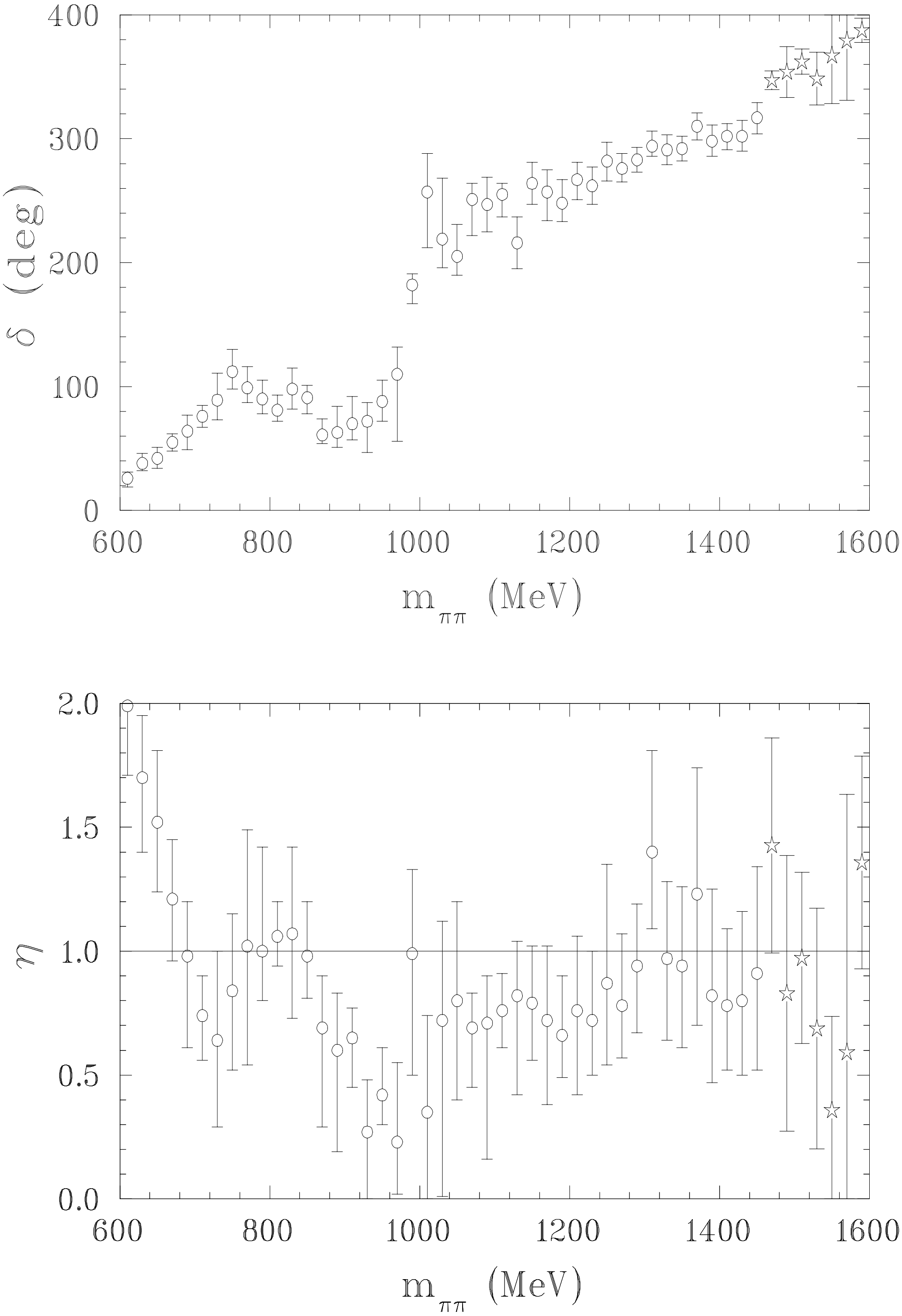


Fig. 5: Phase shifts δ and inelasticities η for the "up–flat" solution (circles). Stars denote data of paper I.

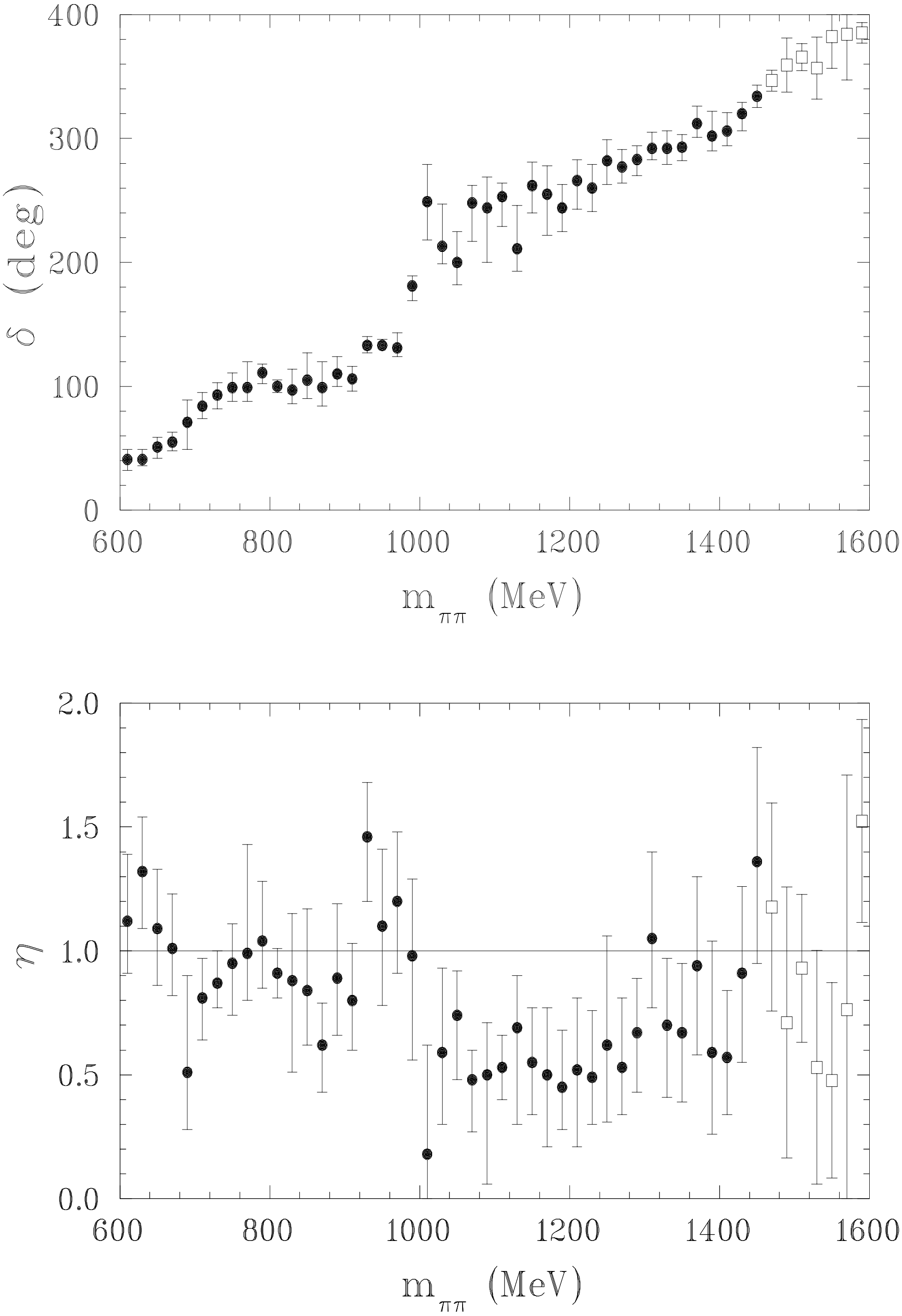


Fig. 6: Phase shifts δ and inelasticities η for the "down–flat" solution (circles). Squares denote data of paper I.

The resulting values of Δ are compared in Fig. 4 with the ansatz of paper I. The variation of Δ as a function of the effective mass is remarkably rapid for the "up" case between 700 and 900 MeV. The values of Δ drop suddenly from about 100^o to negative values as low as -40^o. This jump is related to a considerable difference between the I_0 intensities seen in the same mass range in Fig. 3. On the other hand for the "down" case the values of Δ are positive and do not vary too much in that range. They are also closer to our educated guess on the Δ made in paper I. At 990 MeV the value of Δ cannot be determined for both solutions since the value of I_0 is particularly low at this effective mass (see Fig. 3). The same difficulty is found for two points of $m_{\pi\pi}$ greater than 1420 MeV and smaller than 1460 MeV. Also at 610 MeV for the "up–flat" solution the value of I_0 is too large to satisfy Eqs. (26), (27) and (30), so at all the above masses there are no corresponding points in Fig. 4.

Having new f (the same for each mass bin) and new Δ (different in each bin) we proceed with the calculation of four quantities, above all the phase shifts δ and the inelasticities η of the isoscalar $\pi\pi \rightarrow \pi\pi$ reaction amplitude, as well as the phases and moduli of the a_1 exchange amplitude. In each mass bin we demand them to describe five experimental quantities i.e. the $\pi^0\pi^0$ S-wave intensity in addition to four $\pi^+\pi^-$ observables [10] used in paper I. The fitting program finds solution and errors in each bin, usually with a very small χ^2. The exceptions are 990 MeV, 1430 MeV and 1450 MeV bins for both solutions and 610 MeV for the "up" case. Here the χ^2 is larger (~ 0.1) but reasonable. Above 1460 MeV the fits are not acceptable as it can be expected from Fig. 3.

The results for the most important parameters i.e. the phase shift δ and the inelasticity η are shown in Fig. 5 and Fig. 6. Unfortunately an interesting region around 1500 MeV is hardly covered by this analysis because of the high mass problem in the E852 data. Therefore for $m_{\pi\pi}$ larger than 1460 MeV we show the results of paper I.

For the "up–flat" solution the inelasticity η significantly deviates from unity for $m_{\pi\pi} = 980$ MeV being too large below 700 MeV and too small above 850 MeV. This behaviour of η is unphysical since there are no known inelastic processes which can couple to the $\pi\pi$ channel (see paper II). Also the phase shift δ decreases substantially between 750 MeV and 900 MeV. The unphysical behaviour of η and δ around 900 MeV reflects the large deviation of $\pi^0\pi^0$ data from the predictions based on the $\pi^+\pi^-$ "up-flat" solution – see Fig. 3. Thus a change of Δ or f cannot save the "up–flat" solution once the $\pi^0\pi^0$ data are included. This conclusion is independent of the normalization uncertainties.

On the other hand the "down–flat" solution behaves in a reasonable way. Comparing this solution in Fig. 6 with our results from paper I (see Fig. 7a and Fig. 8) we see that the additional constraint of the E852 data brought

[10]They are: S-wave intensity, $|\overline{g}|/|\overline{h}|$, ϑ_g and ϑ_h.

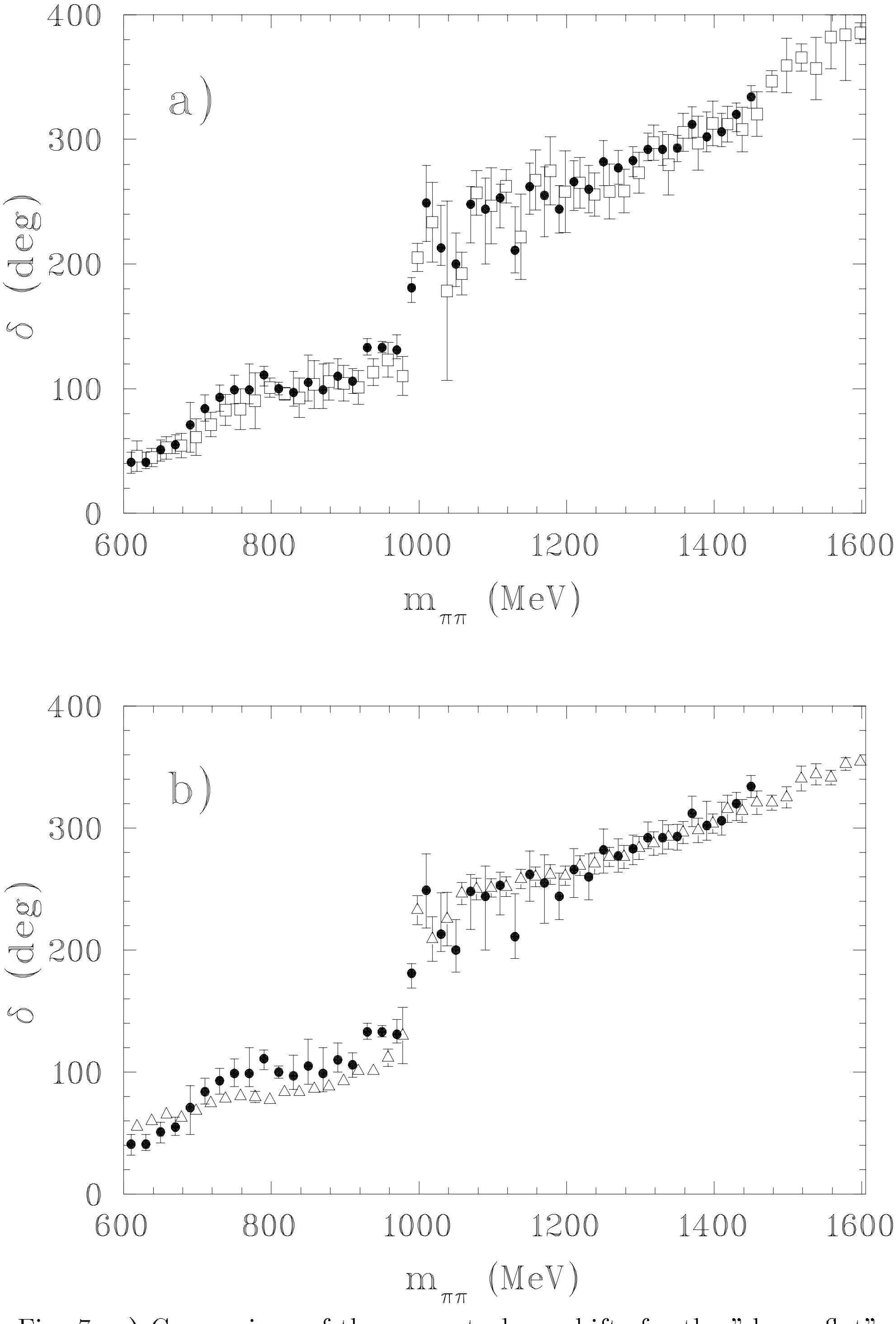


Fig. 7: **a)** Comparison of the present phase shifts for the "down–flat" solution (circles) with the corresponding phase shifts of paper I (squares - shifted by 8 MeV to the right); **b)** similar as in a) but triangles - shifted by 8 MeV to the right - denote the results of analysis B in [1].

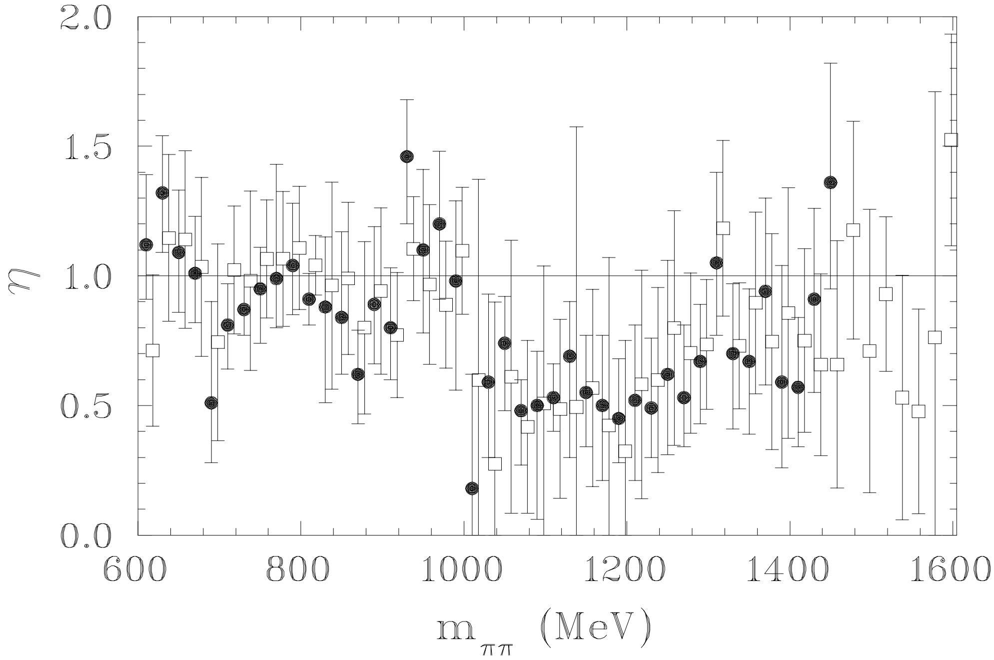


Fig. 8: Comparison of the inelasticities for the "down–flat" solution (circles) with the corresponding ones from paper I (squares - shifted by 8 MeV to the right)

a noticeable reduction of errors. In turn comparing the present solution with the standard one of Ref. [1] we note some differences in spite of an overall similarity (see Fig. 7b). They are clearly seen in the low mass range. The present solution starts at lower δ, then crosses the old one around 700 MeV and stays above it up to the rapid change around 990 MeV.

We have estimated systematical errors due to our particular choice of phases ϑ_g and ϑ_h in Eqs. (31) and (32). If one determines the phase ϑ_h as in Eq. (32) with Δ taken from paper I and calculates ϑ_g to reproduce the experimental values of I_0 then the corresponding results for δ and η lie always very well within the errors displayed in Fig. 6. Thus we conclude that these systematic errors are substantially smaller than the errorrs resulting from the joint fits to the $\pi^+\pi^-$ and $\pi^0\pi^0$data.

Let us finally comment on the role played by the pseudovector (a_1 exchange) transversity amplitudes. These amplitudes are significantly different from zero. Their moduli are shown in Fig. 9.[11] The ratio of the moduli of the pseudovector amplitude to that of pseudoscalar (π exchange) amplitudes fluctuates around $0.2 \div 0.3$ except of the mass regions around the $K\overline{K}$ threshold and around 1500 MeV where moduli of both amplitudes are comparable. In those two regions

[11]Let us note here that in paper I on Fig. 7a) the moduli of the pseudovector amplitudes were incorrectly shown due to replacement of the "down-flat" points by the "up-flat" points.

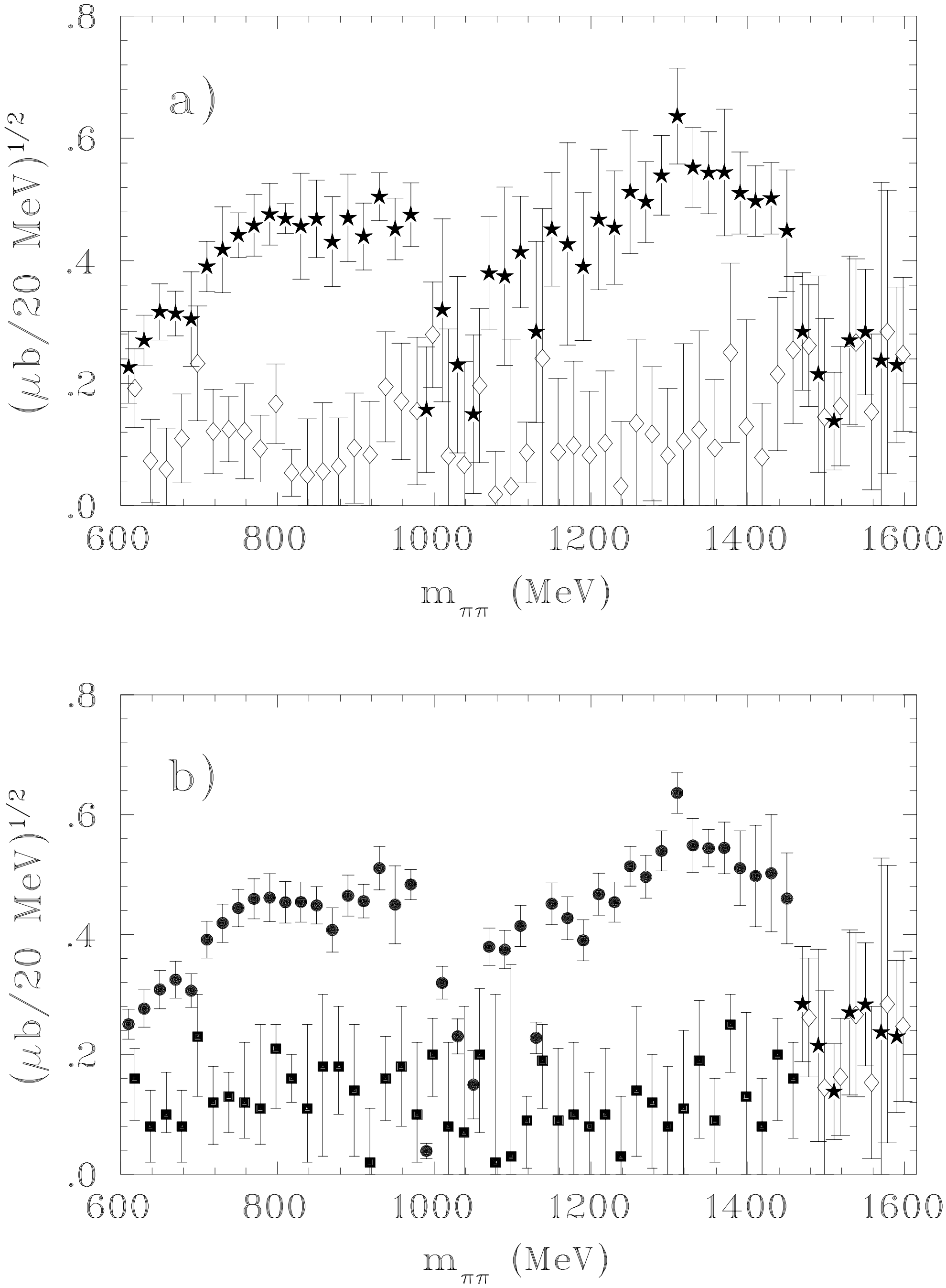


Fig. 9: **a)** Moduli of the pseudoscalar (stars) and pseudovector amplitudes (rhombs - shifted by 8 MeV to the right) for the "down-flat" solution calculated using data of paper I; **b)** Moduli of the pseudoscalar (circles) and pseudovector amplitudes (squares) for the "down-flat" solution of this paper. The points above 1460 MeV are the same as in a).

there exist two scalar resonances $f_0(980)$ and $f_0(1500)$. A closer insight into an interplay of the a_1 and the π exchange amplitudes in the effective mass distribution reveals particularly small values of the interference terms. This result may be the main reason while our more sophisticated analysis has produced the solution for the isoscalar $S-$wave somehow resembling the old one [1] outside of the $m_{\pi\pi}$ ranges near the $f_0(980)$ and $f_0(1500)$ resonances.

In our opinion these results represent now the best knowledge on the $\pi\pi$ S–wave in the mass range $m_{\pi\pi} = (600 \div 1460)$ MeV. They are based on the best $\pi^0\pi^0$ and $\pi^+\pi^-$ data and use the weakest assumptions about the $\pi\pi$ amplitudes.

5 Summary

We have compared the $\pi^+\pi^-$ S–wave production data obtained at 17.2 GeV/c by the CERN-Cracow-Munich collaboration with the $\pi^0\pi^0$ data recently collected by the E852 group. This comparison has been made in two steps. In the first step we have used the one pion and a_1 exchange model developed in paper I to predict the S–wave intensity of the $\pi^0\pi^0$ production at 18.3 GeV/c. Here the previous assumptions about the phases of two independent g and h helicity amplitudes have been directly taken from paper I. Presumably small isotensor part of the a_1 exchange amplitude has been neglected. The proper normalization of the S–wave $\pi^0\pi^0$ intensity has been done using the strong D_0–waves in both experiments. The result of the comparison of the calculated $\pi^0\pi^0$ S–wave intensity with the $\pi^0\pi^0$ data was satisfactory for the "down-flat" solution but the $\pi^0\pi^0$ intensity corresponding to the "up-flat" solution was significantly and systematically too low in the $m_{\pi\pi}$ range between 800 MeV and 1000 MeV.

In the second step we have varied the phase correction Δ to the g and h amplitudes at each $m_{\pi\pi}$ bin trying to fit simultaneously the $\pi^+\pi^-$ and $\pi^0\pi^0$ S–wave intensities. Then we have recalculated the $\pi\pi$ scalar-isoscalar phase shifts and inelasticities. Again for the "down-flat" solution to the $\pi^+\pi^-$ data we have obtained a reasonable behaviour of the inelasticity parameter which, as one should expect, was close to unity below the $K\overline{K}$ threshold. However, the $m_{\pi\pi}$ variation of inelasticity corresponding to the "up-flat" solution was too strong. Also the "up-flat" phase shifts have an unphysical minimum at 900 MeV. This is a clear signal that the "up-flat" solution should be rejected. Since the "down-flat" phase shifts and inelasticities behave regularly we accept the "down-flat" solution, resulting from the joint analysis of the $\pi^+\pi^-$ and $\pi^0\pi^0$ data as a unique physical one.

Acknowledgements
The authors are very grateful to Wolfgang Ochs and Michael Pennington who independently insisted that such an analysis should be done by the Cracow group. This analysis was solely possible thanks to the E852 collaboration which stored the numerical values of their partial waves on the Web thus allowing free access to them. One of us (L.L.) would like to thank Alex Dzierba, Jeffrey Gunter, Adam Szczepaniak and Scott Teige for valuable discussions.

References

[1] G. Grayer *et al., Nucl. Phys.* **B 75**, 189 (1974).

[2] W. Ochs, *Nuovo Cim.* **12 A**, 724 (1972).

[3] P. Estabrooks and A. D. Martin, *Nucl. Phys.* **B 95**, 322 (1975).

[4] B. Hyams *et al., Nucl. Phys.* **B 100**, 205 (1975).

[5] H. Becker *et al., Nucl. Phys.* **B 150**, 301 (1979).

[6] H. Becker *et al., Nucl. Phys.* **B 151**, 46 (1979).

[7] R. Kamiński, L. Leśniak and K. Rybicki, *Zeit. Phys.* **C 74**, 79 (1997).

[8] W. Hoogland *et al., Nucl. Phys.* **B 69**, 266 (1974),
W. Hoogland *et al., Nucl. Phys.* **B 126**, 109 (1977).

[9] M. Svec, A. de Lesquen and L. van Rossum, *Phys. Rev.* **D 46**, 949 (1992),
M. Svec, *Phys. Rev.* **D 53**, 2343 (1996),
M. Svec, *Phys. Rev.* **D 55**, 5727 (1997).

[10] R. Kamiński, L. Leśniak and B. Loiseau, *Eur. Phys. J.* **C 9**, 141 (1999).

[11] R. Kamiński, L. Leśniak and K. Rybicki, *Acta Physica Polonica* **B 31**, 895 (2000).

[12] J. Gunter *et al.* (E852 Coll.), *Phys. Rev.* **D 64**, 07003 (2001).

[13] D. Alde *et al., Zeit. Phys.* **C 66**, 375 (1995).

[14] D. Alde *et al., Eur. Phys. J.* **A 3**, 361 (1998).

[15] W. D. Apel *et al., Nucl. Phys.* **B 201**, 197 (1982).

[16] K. Takamatsu *et al., Prog. Theor. Phys.* **102**, E52 (2001).

[17] B.R. Martin, D. Morgan and G. Shaw, in *Pion-pion interactions in particle physics*, Academic Press, London, 1976.

[18] T.E.O. Ericson, B. Loiseau and A. W. Thomas, Determination of the pion-nucleon coupling constant and scattering lengths, *hep-ph/0009312.*

[19] D. E. Groom *et al.,* Review of Particle Physics, *Eur. Phys. J.* **C 15**, 1 (2000).