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The technique of *inverse Mellin transform* for processes occurring in a *background magnetic field*

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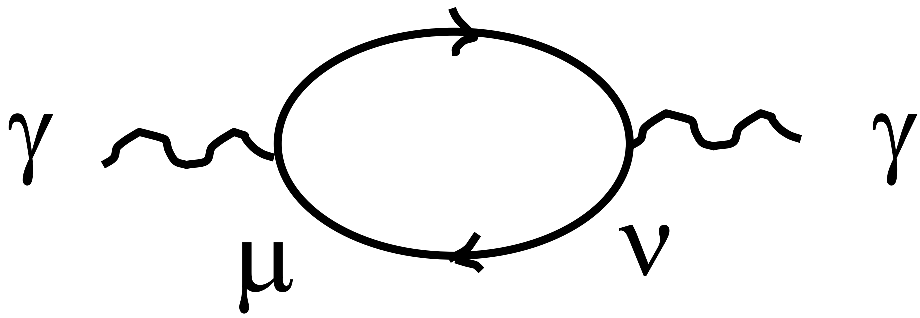
Outline

- Simple illustration-vacuum QED
- Photon polarization function in a background magnetic field
- Conclusion

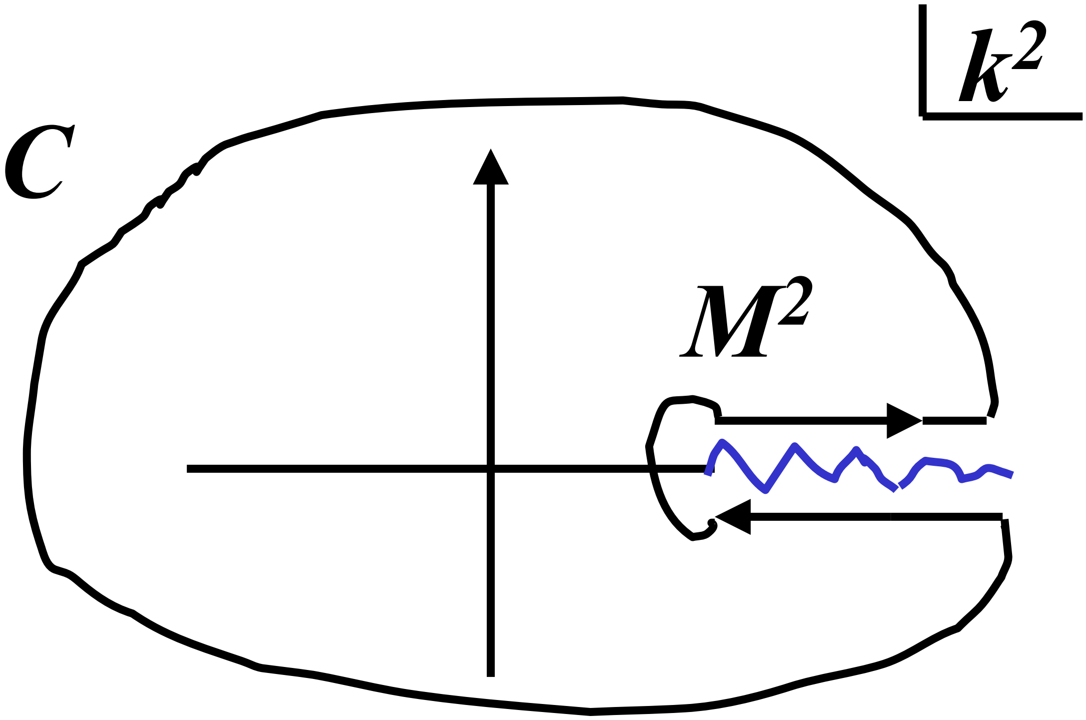
Refs.: W. F. Kao, G.-L. Lin, and J.-J. Tseng, Phys Lett. B 522 (2001) 257;
W. F. Kao and G.-L. Lin, in preparation.

Simple illustration: (QED in the vacuum)



$$i\Pi^{\mu\nu}(k) = (k^2 g^{\mu\nu} - k^\mu k^\nu) i\Pi(k^2)$$

$$\text{Let } \hat{\Pi}(k^2) = \Pi(k^2) - \Pi(0)$$



Consider $I_n = \int_C \frac{dk^2}{2\pi i} \hat{\Pi}(k^2) k^{-2(n+1)}$

One has

$$\frac{1}{n!}\left(\frac{d^n}{d(k^2)^n}\hat{\Pi}(k^2)\right)_{k^2=0}$$

$$= \frac{1}{\pi}\int_{M^2}^{\infty} dk^2 \, \mathrm{Im}\,\hat{\Pi}(k^2) k^{-2(n+1)}$$

$$\frac{1}{n!}\left(\frac{d^n}{dt^n}\hat{\Pi}(t)\right)_{t=0} = \frac{1}{\pi}\int_0^1 du\,\mathrm{Im}\,\hat{\Pi}(u)u^{n-1},$$

with $t = \frac{k^2}{M^2}, u = \frac{1}{t}$.

If we know

$\hat{\Pi}(t) = \sum_{n=0}^{\infty} a(n)t^n$, then

$$a(n) = \frac{1}{\pi}\int_0^1 du\,\mathrm{Im}\,\hat{\Pi}(u)u^{n-1}$$

Hence

$$\mathrm{Im}\hat{\Pi}(u) = \frac{1}{2i}\int_\Gamma ds\,a(s)u^{-s}$$

Inverse Mellin Transform!

In the current case,

$$a(n) = -\frac{2\alpha}{\pi}\frac{\Gamma^2(n+2)}{n\Gamma(2n+4)}\left(\frac{M^2}{m_e^2}\right)^n .$$

Then

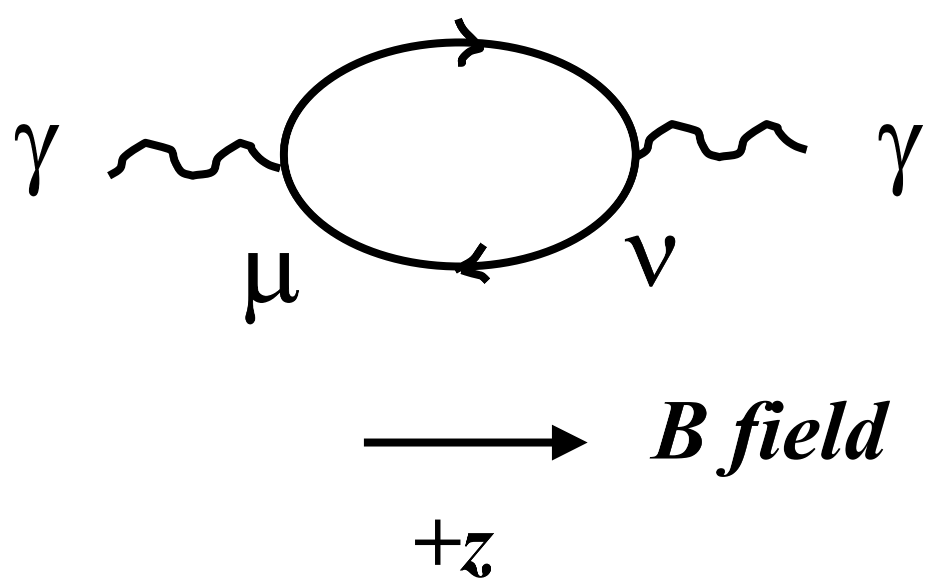
$$\mathrm{Im}\,\hat{\Pi}(u) = -\frac{\alpha}{3}\sqrt{1-\frac{4m_e^2}{M^2}u}\left(1+\frac{2m_e^2}{M^2}u\right)$$
$$\times\Theta\left(1-\frac{4m_e^2}{M^2}u\right),$$

after performing the inverse Mellin transform.

The result is independent of *M*². The inverse Mellin transform generate the correct threshold!

$\mathrm{Re}\,\hat{\Pi}(u)$ is calculable from $\mathrm{Im}\hat{\Pi}(u)$ by the *Kramers – Kronig* relation.

Photon polarization function in a background magnetic field



$$k_{\parallel}^{\mu} = (\omega, 0, 0, k_z), k_{\perp}^{\mu} = (0, k_x, k_y, 0)$$

$\vec{\varepsilon}_{\parallel}$: on the $\vec{B} - \vec{k}$ plane

$\vec{\varepsilon}_{\perp}$: perpendicular to $\vec{B} - \vec{k}$ plane

Schwinger 1951
Tsai and Erber 1974

$$\Pi^{\mu\nu}(k) = \frac{e^3 B}{(4\pi)^2}\int_0^\infty ds \int_{-1}^{+1} dv \{e^{-is\phi_0}[(g^{\mu\nu}k^2 - k^\mu k^\nu)\cdot N_0 -$$

$$(g_\parallel^{\mu\nu}k_\parallel^2 - k_\parallel^\mu k_\parallel^\nu)\cdot N_\parallel + (g_\perp^{\mu\nu}k_\perp^2 - k_\perp^\mu k_\perp^\nu)\cdot N_\perp] -$$

$$e^{-ism_e^2}(1-v^2)(g^{\mu\nu}k^2 - k^\mu k^\nu)\},$$

with $z = eBs$,

$$\phi_0 = m_e^2 + \frac{1-v^2}{4}k_\parallel^2 + \frac{\cos zv - \cos z}{2z\sin z}k_\perp^2$$

$$N_0 = \frac{\cos zv - v\cot z \sin zv}{\sin z},$$

$$N_\parallel = -\cot z(1 - v^2 + \frac{v\sin zv}{\sin z}) + \frac{\cos zv}{\sin z},$$

$$N_\perp = -\frac{\cos zv - v\cot z\sin zv}{\sin z} + 2\frac{\cos zv - \cos z}{\sin^3 z}.$$

Eigenvalue equations for polarization states:

$$\varepsilon_{\parallel}^{\mu} : k^2 + \Pi_{\parallel} = 0,$$

$$\varepsilon_{\perp}^{\mu} : k^2 + \Pi_{\perp} = 0, \text{ with}$$

$$\Pi_{\parallel} = \varepsilon_{\parallel}^{\mu} \Pi_{\mu\nu} \varepsilon_{\parallel}^{\nu}, \Pi_{\perp} = \varepsilon_{\perp}^{\mu} \Pi_{\mu\nu} \varepsilon_{\perp}^{\nu}.$$

Real part of the Eqs.

⇨ **Photon indices of refraction**

Kramers-Kronig relation

Imaginary part of $\Pi_{\parallel,\perp}$

⇨ **Relevant to the photon absorption.**

Take $\Pi_{\parallel}$ as an example :

$$\Pi_{\parallel} = -\frac{\alpha\omega^2 \sin^2\theta}{4\pi}\int_0^{\infty} dz \int_{-1}^{+1} dv \exp[-is\phi_0] N_{\parallel},$$

with $z = eBs$, and

$$\phi_0 = m_e^2 + \frac{1-v^2}{4}k_{\parallel}^2 + \frac{\cos zv - \cos z}{2z\sin z}k_{\perp}^2$$

$$N_{\parallel} = \frac{\cos zv}{\sin z} - \cot z\left(1 - v^2 + \frac{v\sin zv}{\sin z}\right)$$

$$\Downarrow$$

$$\Pi_{\parallel}^{A}$$

Due to the oscillatory behaviors of trigonometric functions, the calculation of $\Pi_{\parallel}$ is non-trivial unless $k_{\parallel}^2 < 4m_e^2$. In this case, one can rotate the contour $z \rightarrow -iz$.

Write

$$\Pi_{\parallel}^{A} = -\frac{\alpha\omega^2 \sin^2\theta}{4\pi}\sum_{0}^{\infty} a(n,k_{\perp}^2)\left(\frac{k_{\parallel}^2}{4m_e^2}\right)^n .$$

$a(n,k_{\perp}^2) \to a(s,k_{\perp}^2)$

s : complex number

Performing inverse Mellin transform on $a(s,k_{\perp}^2)$, we arrive at

$$\mathrm{Im}\,\Pi_{\parallel}^{A} = \frac{2\alpha eB\omega^2 \sin^2\theta}{k_{\parallel}^2} \sum_{l_1=1,l_2=0}^{\infty} T_{l_1 l_2}^{A}\left(\frac{k_{\perp}^2}{eB}\right) \times \frac{\Theta\left(1-\frac{4\lambda eB}{k_{\parallel}^2}+\left(\frac{2\rho eB}{k_{\parallel}^2}\right)^2\right)}{\sqrt{1-\frac{4\lambda eB}{k_{\parallel}^2}+\left(\frac{2\rho eB}{k_{\parallel}^2}\right)^2}},$$

with $\lambda = l_1 + l_2 + \frac{m_e^2}{eB}, \rho = l_1 - l_2.$

***l_1,l_2* are Landau Levels of electron and positron states.**

$\mathrm{Re}\,\Pi_{\parallel}^{A}$ is calculable from $\mathrm{Im}\,\Pi_{\parallel}^{A}$ by the *Kramers – Kronig* relation.

Conclusion

- **We have illustrated the technique of inverse Mellin transform with 2 examples.**
- **With the technique, the photon polarization function in a background magnetic field was calculated.**
- **The infinite number of thresholds for electron-positron production is generated automatically.**

- **The generalizations of this technique to other processes in a background magnetic field are straightforward.**