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LARGE N CALCULATIONS OF PECULIAR PHASE TRANSITIONS IN A SUPERSYMMETRIC MODEL

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Supersymmetric models in the large N limit

d=3, $O(N)$ invariant supersymmetric action

$$\mathcal{S}(\Phi)=\int \mathrm{d}^3x\,\mathrm{d}^2\theta\left[\tfrac{1}{2}\bar{\mathrm{D}}\Phi\cdot\mathrm{D}\Phi+NU(\Phi^2/N)\right]$$

Φ is an N-component vector, $\Phi(\theta)=\varphi+\bar{\theta}\psi+\frac{1}{2}\bar{\theta}\theta F$.

In component form for a generic super-potential:

$$\mathcal{S}=\int \mathrm{d}^3x\,\left[-\tfrac{1}{2}\bar{\psi}\not{\partial}\psi+\tfrac{1}{2}(\partial_\mu\varphi)^2-\tfrac{1}{2}U'(\varphi^2/N)\bar{\psi}\psi\right.$$
$$\left.-U''(\varphi^2/N)(\bar{\psi}\varphi)(\varphi\psi)/N+\tfrac{1}{2}\varphi^2U'^2(\varphi^2/N)\right].$$

Large N methods for supersymmetric actions

Introduce two new superfields:

$$L(\theta) = M + \bar\theta\ell + \tfrac{1}{2}\bar\theta\theta\lambda \qquad R(\theta) = \rho + \bar\theta\sigma + \tfrac{1}{2}\bar\theta\theta s.$$

Add an extra term to the action:

$$\mathcal{S}(\Phi, L, R) = \int \mathrm{d}^3x\, \mathrm{d}^2\theta \left\{ \tfrac{1}{2}\bar{\mathrm{D}}\Phi\cdot \mathrm{D}\Phi + NU(R) + L(\theta)\left[\Phi^2(\theta) - NR(\theta)\right]\right\}$$

Integrate $N-1$ components of Φ, keep $\Phi_1 \equiv \phi$.

$$\mathcal{Z} = \int [\mathrm{d}\phi][\mathrm{d}R][\mathrm{d}L]\, \mathrm{e}^{-\mathcal{S}_N(\phi,R,L)}$$

$$\mathcal{S}_N = \int \mathrm{d}^3x\, \mathrm{d}^2\theta \left[\tfrac{1}{2}\bar{\mathrm{D}}\phi\mathrm{D}\phi + NU(R) + L\left(\phi^2 - NR\right)\right] + \tfrac{1}{2}(N-1)\mathrm{Str}\, \ln\left[-\bar{\mathrm{D}}\mathrm{D} + 2L\right].$$

Note action $\sim N$ and thus three addle point equations

Action density: $\mathcal{E} = \mathcal{S}_N/\text{volume}$:

$\mathcal{E}$ simplifies into

$$\mathcal{E}/N = \tfrac{1}{2} M^2 \varphi^2/N + \frac{1}{24\pi}(m - |M|)^2 (m + 2|M|)$$

$\mathcal{E}$ is positive for all saddle points

$\mathcal{E}$ has an absolute minimum at $m_\varphi \equiv m = |M| = m_\psi$

(a supersymmetric ground state).

Φ^4 super-potential in $d = 3$: phase structure

$$U(R) = \mu R + \tfrac{1}{2} u R^2$$

Gap equations (saddle point equations) reduce to

$$M = \mu - \mu_c + u\frac{\varphi^2}{N} - \frac{u}{4\pi}|M| \,, \quad M\varphi = 0$$

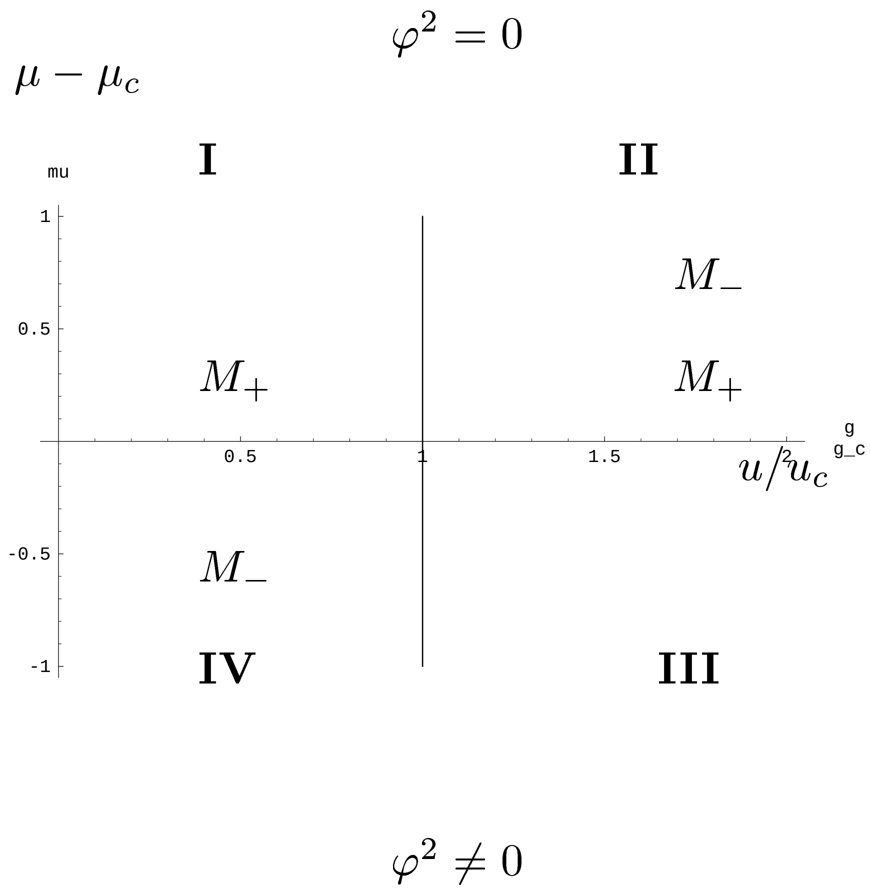

Fig. 1 Summary of the phases of the model in the $\{\mu - \mu_c, u\}$ plane. Here $m_\varphi = m_\psi = |M_\pm| = (\mu - \mu_c)/(u/u_c \pm 1)$. The lines $u = u_c$ and $\mu - \mu_c = 0$ are lines of first and second order phase transitions.

The action density.

$$
\begin{aligned}
\frac{1}{N}\mathcal{E}(m,\varphi) &= \tfrac{1}{2}M^2(m,\varphi)\frac{\varphi^2}{N} \\
&\quad + \frac{1}{24\pi}\left[m - |M(m,\varphi)|\right]^2 \times \left(m_\varphi + 2\,|M(m,\varphi)|\right).
\end{aligned}
$$

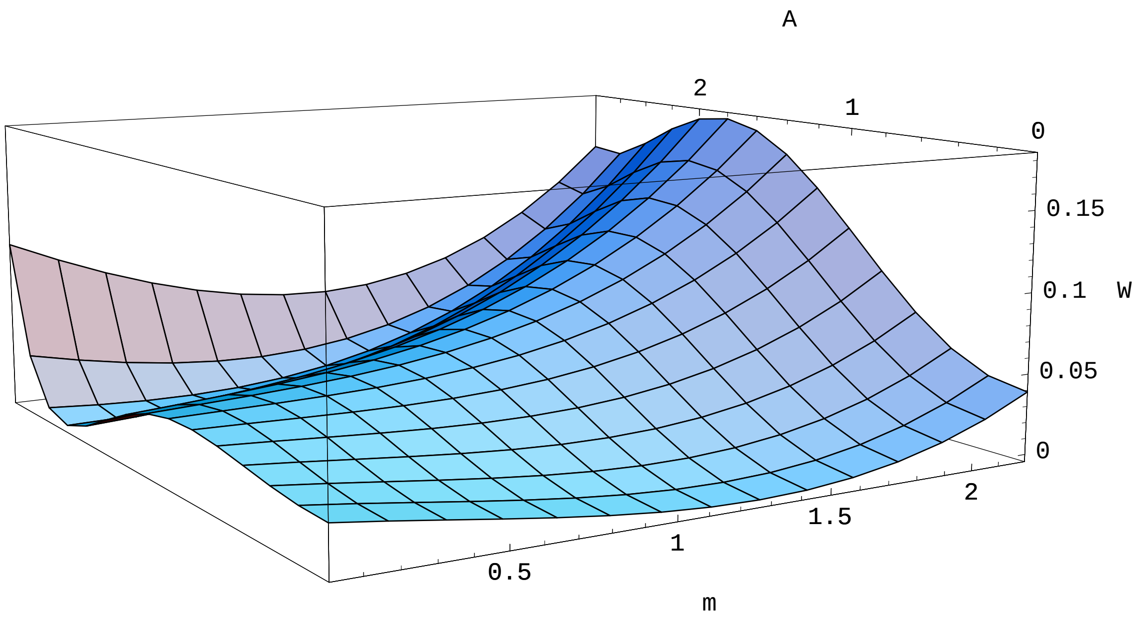

Fig. 2 **Region IV :** The action density $\frac{1}{N}\mathcal{E}(m,\varphi)$ as a function of the boson mass (m) and A, where $A^2 = \varphi^2/u_c$. Here $\mu - \mu_c = -1$, $u/u_c = 0.2$. As seen here there are two distinct degenerate phases. One is an ordered phase $(\varphi \neq 0)$ with a massless boson and fermion, the other is a symmetric phase $(\varphi = 0)$ with a massive $(m = |M_-|)$ boson and fermion.

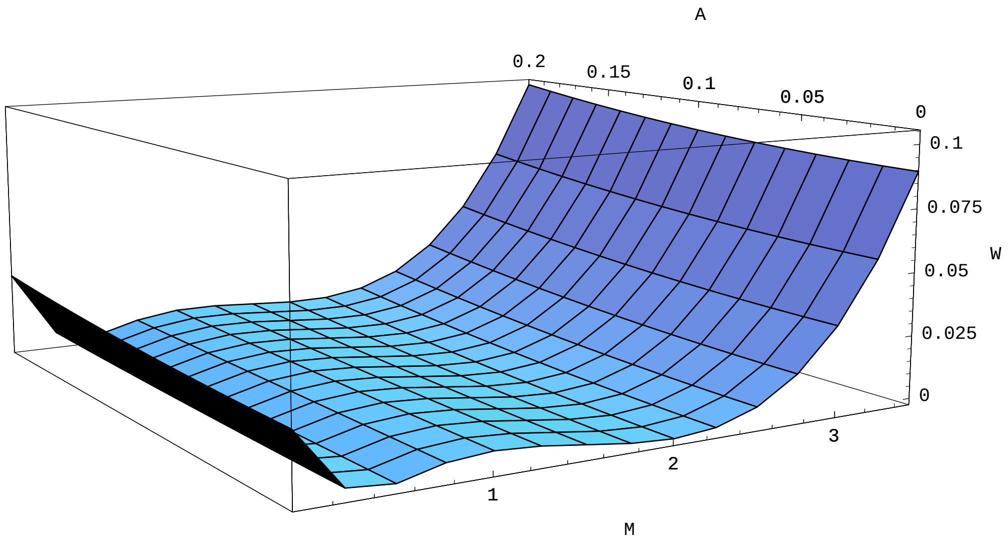

Fig. 3 **Region II:** The action density $\mathcal{E}(m, \varphi)$ as a function of the boson mass (m) and A, where $A^2 = \varphi^2/u_c$. Two degenerate, $O(N)$ symmetric phases exist with massive bosons (and massive fermions). Here $\mu - \mu_c = 1$ (sets the mass scale) $u/u_c = 1.5$.

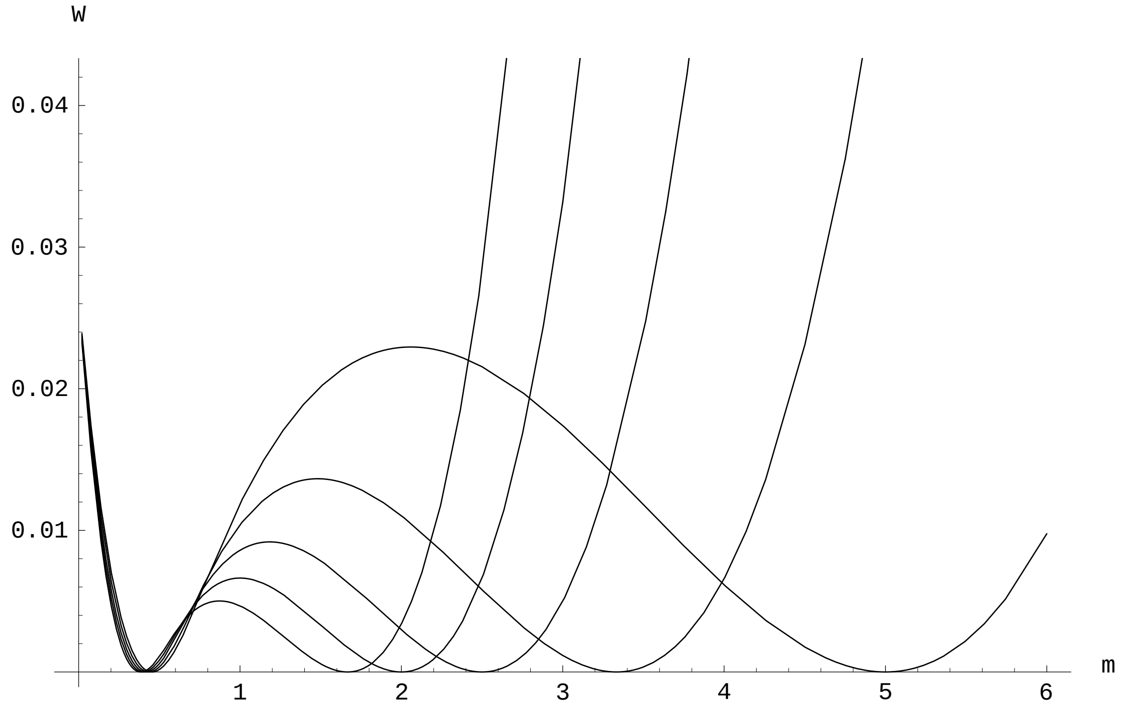

Fig. 4 The action density $\frac{1}{N}\mathcal{E}(m, \varphi = 0)$ in region **II** of Fig. 1. Here $\mu - \mu_c = 1$ (sets the mass scale) and u/u_c= varies between 1.6 and 1.2. There are two massive, degenerate $O(N)$ symmetric SUSY vacua with $\varphi = 0$ and possible large hierarchy of masses $|M_-/M_+| >> 1$ (as $u \to u_c$) with $m_\psi = m_\varphi = M_+ = (\mu - \mu_c)/(u/u_c + 1)$ and $m_\psi = m_\varphi = -M_- = (\mu - \mu_c)/(u/u_c - 1)$.

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Special situation.

$$u = u_c = 4\pi \, .$$

0.1 $\langle LL \rangle$ propagator, bound states and massless fermion and boson $O(N)$ singlet bound states

The $\langle LL \rangle$

$$-\frac{N}{2u}\int \mathrm{d}^3x\,\mathrm{d}^2\theta (L-\mu)^2 + \tfrac{1}{2}(N-1)\mathrm{Str}\,\ln\left(-\bar{\mathrm{D}}\mathrm{D}+2L\right).$$

$$\Delta_L^{-1} = -\frac{N}{4\pi|M|}\left[1+\{M+|M|(u_c/u)\}\delta^2(\theta'-\theta)\right]\mathrm{e}^{i\bar{\theta}\not{p}\theta'}\, .$$

Corresponds to a super-particle of mass $2M(1+u_c/u)$ At the special point $u=u_c$ the mass vanishes.
E.g. the $\psi\varphi$ scattering amplitude $T_{\psi\varphi,\psi\varphi}(p^2)$, in the limit $p^2 \to 0$ satisfies

$$T_{\psi\varphi,\psi\varphi}(p^2) \sim \frac{2u}{N}\left[1+\frac{u}{4\pi}\frac{M}{|M|}+\frac{u}{2\pi}\frac{i\not{p}}{|M|}\right]^{-1} \to -\frac{4\pi i}{N}\frac{|M|}{\not{p}}$$

for $M<0$ and $u \to u_c$

A supersymmetric non-linear σ-model at large N

Consider the supersymmetric non-linear σ-model in d dimensions, $2 \le d \le 3$.

$$\mathcal{S}(\Phi) = \frac{1}{2\kappa} \int \mathrm{d}^d x \, \mathrm{d}^2\theta \, \bar{\mathrm{D}}\Phi \cdot \mathrm{D}\Phi \tag{0.1}$$

Dimension $d = 3$.

$$\mathcal{E}/N = \frac{1}{2N\kappa} M^2 \varphi^2 + \frac{1}{24\pi} (m - |M|)^2 (m + 2|M|), \tag{0.2}$$

Dimension $d = 2$.

$$\frac{\mathcal{E}}{N} = \frac{1}{8\pi} \left[m^2 - M^2 + 2M^2 \ln(M/m) \right].$$

$$M = m = \Lambda \, \mathrm{e}^{-2\pi/\kappa} \, .$$

$O(N)$ supersymmetric model at finite temperature

$$\mathcal{Z} = \int [\mathrm{d}\phi][\mathrm{d}R][\mathrm{d}L]\,\mathrm{e}^{-\mathcal{S}_N(\phi,R,L)}, \tag{0.3}$$

$$\begin{aligned}\mathcal{S}_N = \int \mathrm{d}^3x\,\mathrm{d}^2\theta\,\left[\tfrac{1}{2}\bar{\mathrm{D}}\phi\mathrm{D}\phi + NU(R) + L\left(\phi^2 - NR\right)\right] \\ + \tfrac{1}{2}(N-1)\mathrm{Str}\,\ln\left[-\bar{\mathrm{D}}\mathrm{D} + 2L\right].\end{aligned} \tag{0.4}$$

$$\Delta(k,\theta,\theta) = \frac{1}{k^2+M_T^2+\lambda} + \bar\theta\theta M_T\left(\frac{1}{k^2+M_T^2+\lambda} - \frac{1}{k^2+M_T^2}\right)$$

M_T the expectation value of $M(x)$ at finite temperature T.

$$\begin{aligned}G_2(m_T,T) &= \frac{T}{(2\pi)^{d-1}}\sum_{n\in\mathcal{Z}}\int^\Lambda \frac{\mathrm{d}^{d-1}k}{(2\pi nT)^2+k^2+m_T^2} \\ &= \int^\Lambda \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\frac{1}{\omega(k)}\left(\frac{1}{2}+\frac{1}{\mathrm{e}^{\beta\omega(k)}-1}\right)\end{aligned}$$

Fermions:

$$\mathcal{G}_2(M_T,T) = \int^\Lambda \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\frac{1}{\omega(k)}\left(\frac{1}{2}-\frac{1}{\mathrm{e}^{\omega(k)/T}+1}\right)$$

with $\omega(k) = \sqrt{k^2+M_T^2}$.

$$\frac{1}{N}\mathcal{F} = -\frac{F^2}{2N} + M_T\frac{F\varphi}{N} + \lambda\frac{\varphi^2}{2N} + \tfrac{1}{2}s(U'(\rho) - M_T) - \frac{1}{12\pi}\left(m_T^3 - |M_T|^3\right)$$
$$+ \tfrac{1}{2}\lambda(\rho_c - \rho) + T\int\frac{\mathrm{d}^2k}{(2\pi)^2}\left\{\ln[1 - \mathrm{e}^{-\beta\omega_\varphi}] - \ln[1 + \mathrm{e}^{-\beta\omega_\psi}]\right\}$$

Peculiar transitions can occur in this system

Region **(II)** $\mu - \mu_c \geq 0$ and $u \geq u_c$: (Fig. 5).

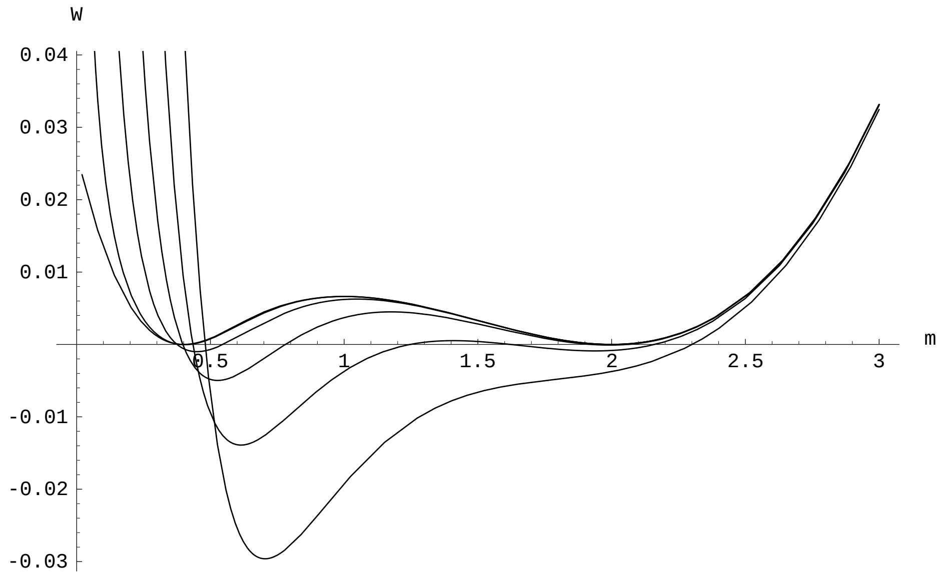

Fig. 5 The ground state free energy $\frac{1}{N}\mathcal{F}(m \equiv m_\varphi, \varphi, T)$ at $\varphi = 0$ as a function of the boson mass (m) at different temperatures. Here $\mu - \mu_c = 1$(this sets the mass scale) , $u/u_c = 1.5$ and T varies between $T = 0 - -0.5$. At $T = 0$ two degenerate $O(N)$ symmetric phases exist with a light $m = m_+$ and heavier $m = m_- > m_+$ massive boson (and fermion). As the temperature increases, the light mass phase is stronger affected as its entropy increases faster and becomes the only ground state.

Region **(IV)** $\mu \leq 0$ and $u \leq u_c$: Fig. 2).

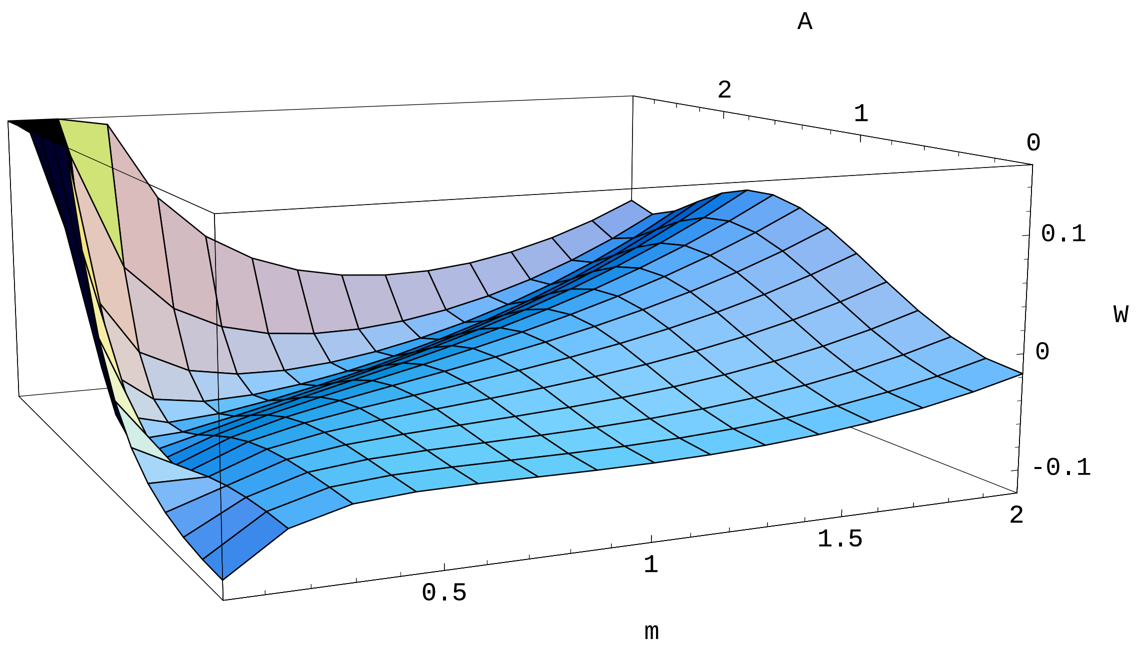

Fig. 6 Same as Fig. 2 but the temperature has been increased from $T = 0$ (in Fig. 2) to $T = 0.7$ (here). $\frac{1}{N}\mathcal{F}(m \equiv m_\varphi, \varphi, T)$ as a function of the boson mass (m) and A, where $A^2 = \varphi^2/u_c$. Here $\mu = -1$ and $u/u_c = 0.2$. A non-degenerate $O(N)$ symmetric ground state ($\varphi = 0$) appears with a very small boson mass (the non-zero mass is not seen here due to the limited resolution of the plot).

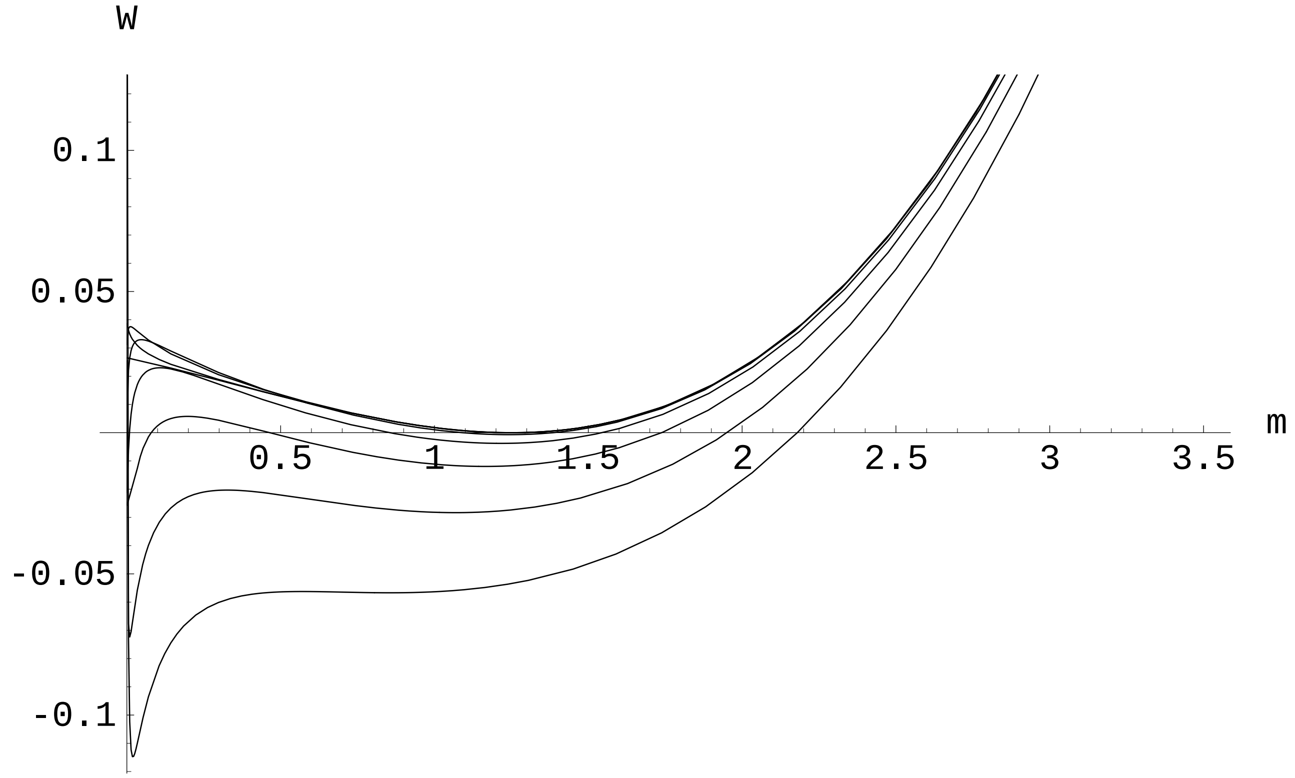

Fig. 7 This figure displays the effect of increasing the temperature from $T = 0$ in Fig. 2 to T that varies between $T = 0 - 0.7$ (Fig. 6 has $T = 0.7$). The ground state free energy $\frac{1}{N}\mathcal{F}(m \equiv m_\varphi, \varphi, T)$ at $\varphi = 0$ is plotted as a function of the boson mass (m) at different temperatures. Here $\mu = -1$, $u/u_c = 0.2$. At $T = 0$ there are two degenerate phases: An $O(N)$-symmetric phase, shown here, with a massive ($m = m_-$) boson and fermion and an ordered phase ($\varphi \neq 0$) with massless particles (both phases are shown in Fig. 2). At finite temperatures the $O(N)$ symmetry is restored (see Fig. 6) and a small mass ground state appears, the heavy mass state decays into the small mass ground state as seen here.

Supersymmetric $O(N)$ non-linear σ-model at finite temperature

$$\mathcal{Z} = \int [\mathrm{d}\Phi][\mathrm{d}L]\,\mathrm{e}^{-\mathcal{S}(\Phi,L)} \tag{0.5}$$

with

$$\mathcal{S}(\Phi) = \frac{1}{2\kappa}\int \mathrm{d}^d x\,\mathrm{d}^2\theta\,\bar{\mathrm{D}}\Phi\cdot\mathrm{D}\Phi + L(\Phi^2 - N)\,, \tag{0.6}$$

The free energy is given by

$$\frac{1}{N}\mathcal{F} = \tfrac{1}{2}(M_T^2 - m_T^2)\left(\frac{1}{\kappa} - \frac{1}{\kappa_c}\right) - \frac{1}{12\pi}\left(m_T^3 - |M_T|^3\right) \\ + T\int \frac{\mathrm{d}^2 k}{(2\pi)^2}\left\{\ln[1-\mathrm{e}^{-\beta\omega_\varphi}] - \ln[1+\mathrm{e}^{-\beta\omega_\psi}]\right\}.$$

(for $\varphi = 0$):

$$\frac{1}{N}\mathcal{F} = \frac{1}{24\pi}(m_T - |M_T|)^2(m_T + 2|M_T|) + \frac{T}{4\pi}(m_T^2 - M_T^2)\ln(1-\mathrm{e}^{-m_T/T}) \\ + T\int \frac{\mathrm{d}^2 k}{(2\pi)^2}\{\ln(1-\mathrm{e}^{-\beta\omega_\varphi}) - \ln(1+\mathrm{e}^{-\beta\omega_\psi})\}.$$

notation

$$X(\kappa, T) = \exp\left[\frac{2\pi}{T}\left(\frac{1}{\kappa_c} - \frac{1}{\kappa}\right)\right].$$

An interesting non-analytic behaviour:

$$\begin{cases} M_T = 0 & \text{for } X < 2, \\ M_T = 2T\ln\left[\frac{1}{2}(X + \sqrt{X^2-4})\right] & \text{for } X > 2\,. \end{cases}$$

The boson thermal mass m_T:

$$m_T = 2T \ln \left[\tfrac{1}{2}(X + \sqrt{X^2+4}) \right].$$

For $\kappa < \kappa_c$ and $T \to 0$ we find the asymptotic behaviour

$$m_T \sim T X(\kappa, T),$$

Dimension $d = 2$. At high temperature

$$\frac{T}{m_T} \sim \frac{1}{\pi} \ln(m_T/m) \sim \frac{1}{\pi} \ln(T/m),$$