The Frontier of Multi-Loop Feynman Diagrams: (Semi-) Numerical Techniques

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Why do we need *fast & efficient* ways to deal with

$\boxed{\textbf{M l \& L (Giga -) Feynman diagrams?}}$

(**SM** $\equiv$ 11 lines, 57 vertices, **modulo** f-multiplicity)

- A **FULL two-loop** EW needed for **GigaZ** ($10^9\,Z$ events);
- **Theoretical Uncertainty** (e.g. BlueBand) not **yet** satisfactory, requires a **full two-loop** analysis of the **SM $\oplus$ Beyond**

$\boxed{\text{a } \textbf{real} \rightarrow \text{ Multi - Scale } \leftarrow \text{ problem}}$

We have been moving from a **mediaeval architecture** of radiative corrections to the **renaissance** of **LEP period** and now one should not indulge in deciphering the intimate meaning of the **last digit** in the **last fit** but prepare for the new era (hope is not **baroque**), **although** many auxiliary features are far from satisfactory,

Diagrammatica	versus	**Numerica**
new algorithms	"	**standardization**
growth in complexity	"	**software problems**

Analytically quasi - solvable cases:

a) **QCD**, all masses $\equiv 0$;

b) **QED**, only $m_e \neq 0$ and 0 or **mass shell** for external scales;

c) **SM** $\oplus$ simplifying conditions, e.g. $m/m_{\mathrm{top}} \ll 1$.

for the **general** $\searrow$ case the **analytical way** not feasible

arbitray int/ext masses

although a lot can be done what is so special about the **AW**???

Analytical properties explicit $\oplus$ an **EXPANSION** available, cf.

$$\mathrm{Li}_2\,(z) \;=\; \sum_{n=0}^{\infty} B_n \frac{\left[-\ln(1-z)\right]^{n+1}}{(n+1)\,!},$$

expansion parameter with the same **cut** of $\mathrm{Li}_2\,(z)$
Beyond **one-loop** and for the **general** mass case **polylogarithms** do not suffice! $\Longrightarrow$

AW is in general possible only for **small** $\#$ of **Scales** (masses and momenta or invariants), cf.

Sunset $\;\equiv\; \sum$ generalized hypergeometric series

i.e. accessible for **small or large** p^2 but not in the **intermediate** region (**Thresholds!!**)

Alternative strategy: Numerical Way (I)

- **Berends, Böhm** *et al*

  Dispersion Relations → one-dimensional **integral representation** for two-loop **self-energies**.

- **Caffo, Czyz** and **Remiddi**

  All diagrams → (limited) number of **MI** ≡ **master integrals MI through** differential equations in p_i^2

  Results available for **sunset**, **difficulty** in determining the **initial conditions** to be provided by other methods.

- **Shimizu** *et al*

  Fully **Numerical** integration below thresholds

  Above thresholds, numerical integration is done after

  a) **Symmetric Method**, the integrand is replaced by a symmetric sum so as to eliminate the singularity;

  b) **Hybrid Method** Integrals for a few variables analytically until an integrable function is obtained,

  Applied to **planar** and **non-planar** two-loop vertices.

□ Soper

Calculations (in **QCD**) are performed using **Numerical** integration over FS particles and

Analytical integration over virtual particles,

it requires to deform some of the integration contours into the complex plane.

□ Ghinculov and Yao

All diagrams $\rightarrow$ generalized sunset

$$\text{Any diagram} \;=\; \int_{\text{ext}} \text{generalized sunset}$$

One-dimensional integral representation for the 10 **generalized**

sunset MI needed in a renormalizable theory

$\oplus$ numerical integration over the **ext** Feynman parameters.

Conclusion:

Size of the problem $\Longrightarrow$ Full *Numerical Calculability* can hardly be avoided. Previous example: $\text{Amp(pol)} \in C$ to be squared numerically.

A **GENERIC** (scalar) N_L - Legs, n_l - Loops diagram

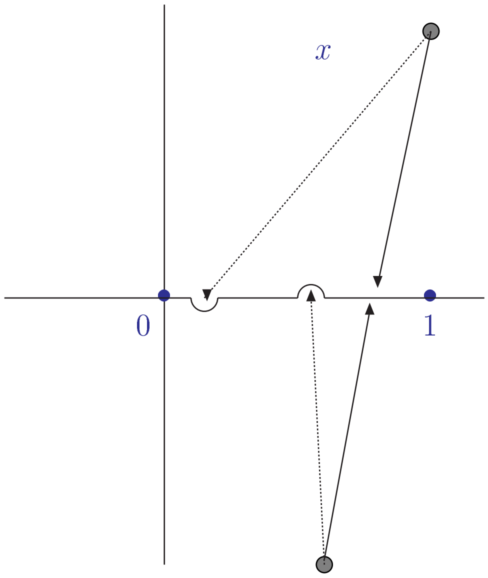
$$G = \left(i\,\pi^{n/2}\right)^{n_l}\,\Gamma\left(N_L - \frac{n}{2}\,n_l\right) \int \frac{dx_G\,\delta\,(1-x_G)}{U^{n/2}\,(V - i\,\delta)^{N_L - n n_l/2}},$$
$$\equiv \int_S \prod_{i=1}^{N_l - 1} dx_i\,I_G(\{x_i\}), \quad dx_G = \prod_{i=1}^{N_l} dx_i, \;\; x_G = \sum_{i=1}^{N_l} x_i.$$

The parametric function I_G has not a **definite sign**, i.e. poles inside the integration region that hinder the **numerical integration**. Example:

$$I(\alpha) = \int_0^1 dx\,[\chi_B(x) - i\,\delta]^{-\alpha}$$
$$\chi_B(x) = a\,x^2 + b\,x + c = a\,(x + \frac{b}{2\,a})^2 - \frac{b^2 - 4\,a\,c}{4a}$$

When the roots of $\chi_B - i\,\delta = 0$ are real and $\in [0,1]$ the integral can be computed by distorting the integration contour unless a pinch will occur; in this case the integral is singular.

pinch singularity



Numerical Way (II) (TOPSIDE Collaboration)

TOrino **P**roject **S**upporting **I**ntensive **D**iagrams **E**valuation: A. Ferroglia (NYU), G. Passarino (Torino), M. Passera (Padova),S. Uccirati (Torino)

The method $\Longrightarrow$ seek algorithm(s) for smoothness that avoid spurious singularites

$$G = \frac{1}{B_G} \int_S dx\, \mathcal{F}[I_G]$$

- $\triangleright$ $\mathcal{F}$ is a functional of I_G that makes the whole integrand smooth in **parametric space** (from **poles** to **cuts**),
- $\triangleright$ B_G is a function of masses and momenta that contains the **TRUE** singularities of G (if any).

Prolegomena: Bernstein-Tkachov theorem:

For any finite set of polynomials $V_i(x)$, $x = (x_1, \ldots, x_N)$ there exists an identity of the following form:

$$\mathcal{P}\,(x, \partial) \prod_i V_i^{\mu_i+1}(x) = B \prod_i V_i^{\mu_i}(x).$$

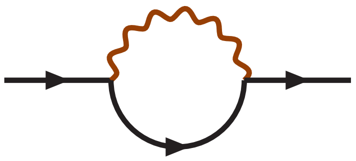
where $\mathcal{P}$ is a polynomial of x , $\partial_i = \partial/\partial x_i$; B and all coefficients of $\mathcal{P}$ are polynomials of μ_i and of the coefficients of $V_i(x)$.

Iterative applications of the functional relations, followed by **integration by parts**, allow us to express the integrand in **parametric space** as a combination of $P^N(x) \times \ln P(x)$

STRATEGY

- **Diagram**

Fermionic self-energy in QED



- **Parametric representation**

$$\Sigma = -ie^2\,\pi^{2-\epsilon/2}\,\mu^\epsilon\,\Gamma\left(\frac{\epsilon}{2}\right)\int_0^1 dx\,\left[i\,(2-\epsilon)\,x\not{p} + (4-\epsilon)\,m_e\right]\chi^{-\epsilon/2},$$

$$\chi(x) = (1-x)\left(x\,p^2+m_e^2\right).$$

- **Reduction to Form - Factors**

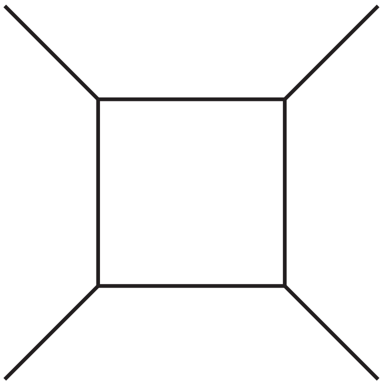
$$\Sigma = -i\pi^2 e^2\,\left[\Sigma_p\,i\not{p} + \Sigma_m\,m_e\right],$$

- **BT $\oplus$ integration - by - parts**

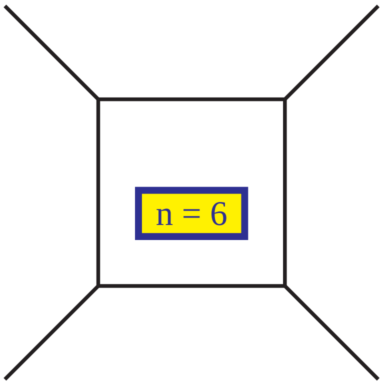
$$\Sigma_p = \frac{1}{\bar{\varepsilon}} + \frac{p^2}{(p^2+m_e^2)^2}\,\Big[\frac{\gamma}{2}\,\frac{(p^2+m_e^2)^2}{p^2} - \frac{1}{3}\,m_e^2 - \frac{1}{6}\,p^2 + \int_0^1 dx\,\left(\frac{1}{2}-\frac{1}{2}\,\frac{m_e^2}{p^2}-4\,x\right)\chi\,\ln\frac{\chi}{\mu^2}\Big], \Longleftarrow$$

$$\Sigma_m = \frac{4}{\bar{\varepsilon}} + \frac{p^2}{(p^2+m_e^2)^2}\,\Big[\left(2\,\gamma - \ln\frac{m_e^2}{\mu^2}+\frac{1}{2}\right)\frac{(p^2+m_e^2)^2}{p^2} + p^2\,\ln\frac{m_e^2}{\mu^2} - \frac{2}{3}\,p^2 + m_e^2\left(3\,\ln\frac{m_e^2}{\mu^2}-2\right) - 6\int_0^1 dx\,\chi\,\ln\frac{\chi}{\mu^2}\Big]. \Longleftarrow$$

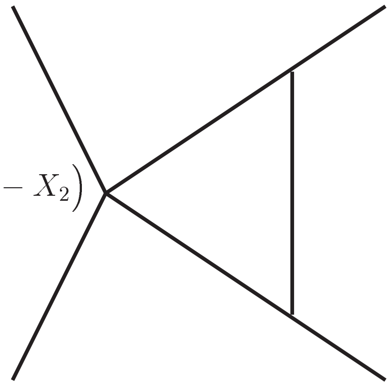
Diagrammatica of BT functional relations:



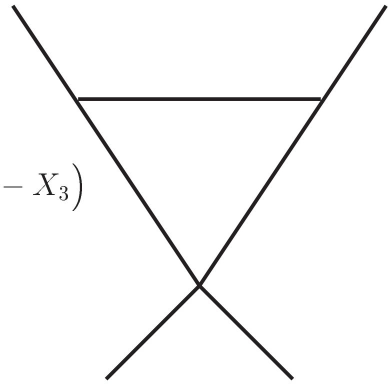
$= \frac{1}{2\,B_4}\Big\{ \; -$



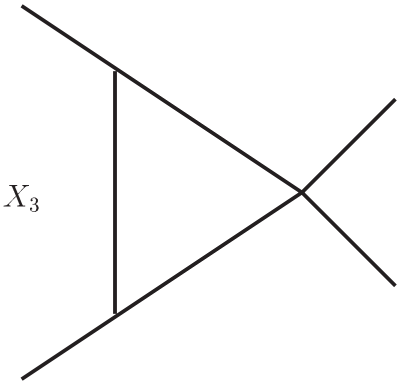
$+ \; \big(X_1 - X_2\big)$



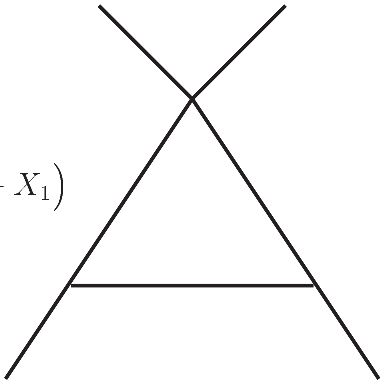
$+ \; \big(X_2 - X_3\big)$



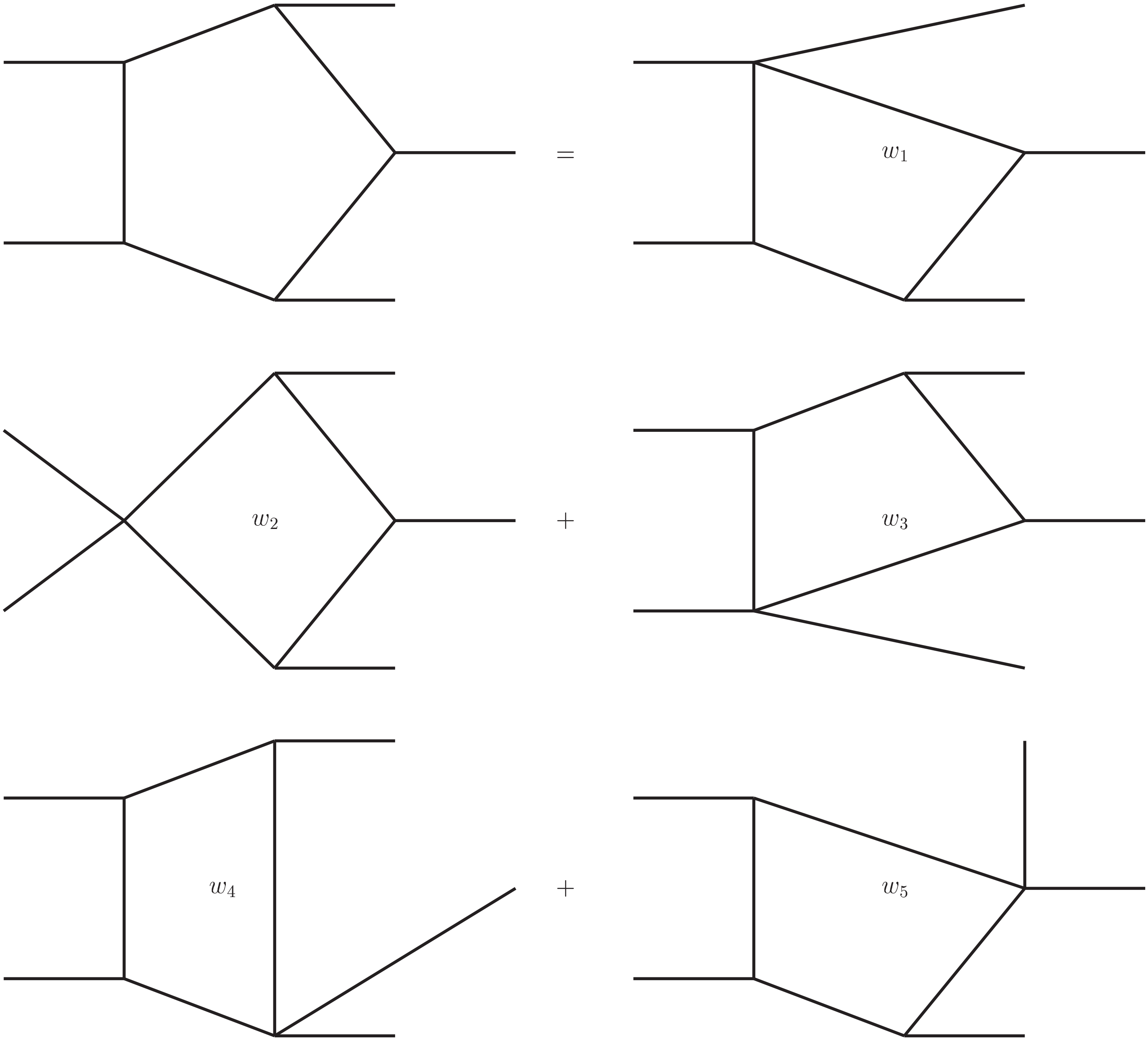
$+ \; X_3$



$+ \; \big(1 - X_1\big)$



$\Big\}$



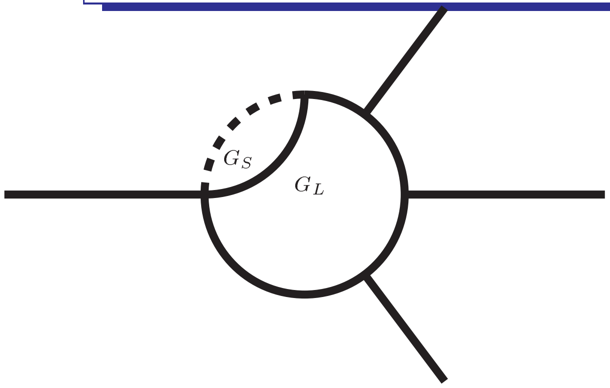
$$
\begin{aligned}
w_1 &= \frac{1}{2}\,\frac{1-X_1}{B_5}, \quad w_2 = \frac{1}{2}\,\frac{X_1-X_2}{B_5}, \quad w_2 = \frac{1}{2}\,\frac{X_2-X_3}{B_5}, \\
w_4 &= \frac{1}{2}\,\frac{X_3-X_4}{B_5}, \quad w_5 = \frac{1}{2}\,\frac{X_4}{B_5},
\end{aligned}
$$

Solution for the BT - polynomial $\mathcal{P}$ known $\forall$ **All Quadrics**

$\equiv$ **all** $1l$ & nL diagrams[1]

Sunset topology (**the easiest**) requires **degree** $= 4$. Furthermore, **around - threshold** behavior remains unsolved in **BT** approach,

Minimal BT -algorithm



$$G = \int dx_S \frac{1}{B(x_S)} \int dx_L \, \mathcal{F}[I_G(x_S, x_L)].$$

Analytical structure of G_S simpler, x_S integration with **contour - distortion**, always possible **unless** a **pinch** (or an **end-point**) singularity occur $\cdots\cdots$ but this is a real (Landau) singularity. However,

$\forall$ two-loop two-point functions **2l&2L**

$\forall$ two-loop three-point functions **2l&3L** (but one)

contour - distortion is not needed $\Longrightarrow$

$\forall$ of them a **parametrization** can be found that allows for a **BT** procedure with $B \equiv B(\textbf{parameters})$. These $2l$ diagrams are no more involved that a $1l$ diagram and require Normal BT $\oplus$ integration by parts. $B = 0 \;\rightarrow$ **non-leading** LS.

[1] Work in progress towards algebraic solution for B_G / numerical for $\mathcal{P}$

Why a difficult problem?

Numerica

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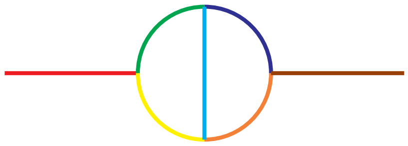
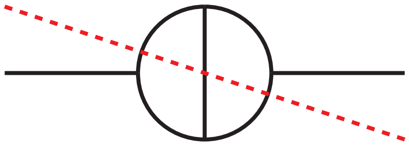
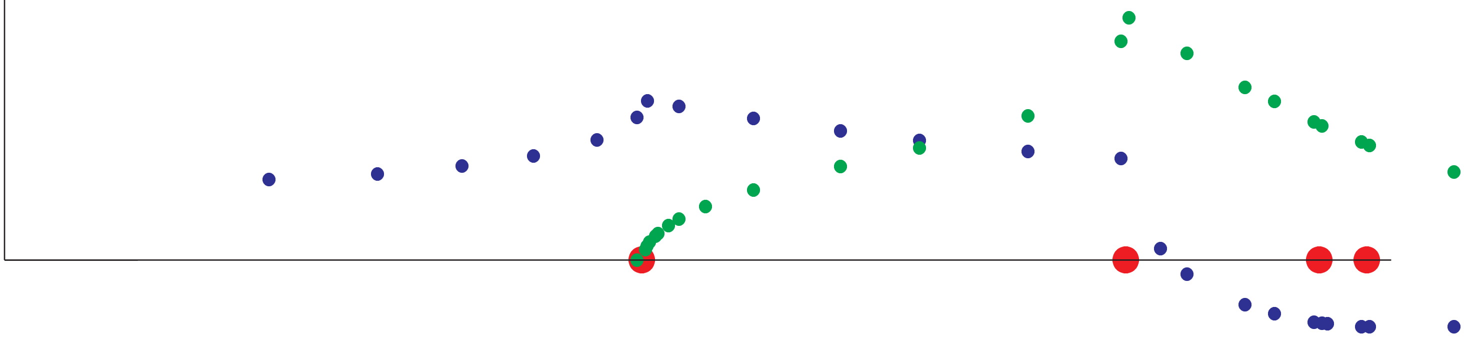
By TOPSIDE.F





$+ \quad \cdots\cdots$



Numerical Evaluation requires **Precise** knowledge of analytical structure, i.e. **Solution** of $\boxed{\textbf{Landau - Bjorken}}$ equations[2].

$$C_0 \searrow$$

leading singularity $\equiv B_3 = 0 \equiv$ anomalous threshold (**AT** $\to A_{\rm abs}$ is not directly related through **unitarity** to physical processes)

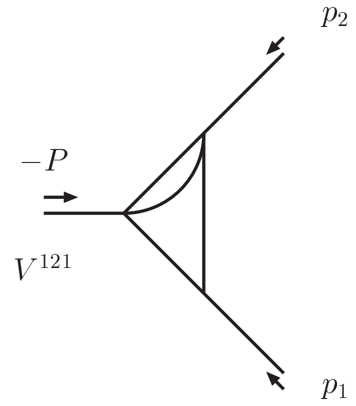
$$\begin{aligned}
s &= \frac{1}{2\,m_2^2}\Big\{2\,m_2^2\,(m_1^2+m_3^2) \\
&- (m_1^2+m_2^2-M_1^2)\,(m_3^2+m_2^2-M_2^2) \\
&\pm \Big[\lambda(M_1^2,m_1^2,m_2^2)\lambda(M_2^2,m_3^2,m_2^2)\Big]^{1/2}\Big\},
\end{aligned}$$

$$(p_1+p_2)^2 = -s, \qquad p_i^2 = -M_i^2.$$

sub-leading singularites $\equiv$ zeros of

$$\lambda(s,m_1^2,m_3^2), \quad \lambda(M_1^2,m_1^2,m_2^2), \quad \lambda(M_2^2,m_2^2,m_3^2).$$

$\equiv$ normal (cuts) & pseudo thresholds.



$$\begin{aligned}
P^2 &= \frac{1}{2\,m_3^2}\Big[(-p_2^2-(m_1+m_2)^2+m_3^2)\,(-p_1^2+m_3^2-m_4^2) \\
&+ 2\,m_3^2\,(p_1^2+p_2^2) \pm (\lambda_1\,\lambda_2)^{1/2}\Big]
\end{aligned}$$

$$\lambda_1 = \lambda(-p_1^2,m_3^2,m_4^2) \qquad \lambda_2 = \lambda(-p_2^2,(m_1+m_2)^2,m_3^2)$$

[2]In principle also non-Landau or second-type singularities should be considered

Around $B_G = 0$

- $B_3 \neq 0, \qquad V = (x - X)^t\, H\, (x - X) + B_3$

$$\begin{aligned} C_0 &= \frac{1}{B_3}\{\frac{1}{2} + \int_0^1 dx_1\, [\int_0^{x_1} dx_2\, \ln V(x_1, x_2) \\ &+ \frac{1}{2}(X_2 - X_1)\, \ln V(x_1, x_1) - \frac{1}{2} X_2\, \ln V(x_1, 0) \\ &- \frac{1}{2}(1 - X_1)\, \ln V(1, x_1)]\}. \end{aligned}$$

- $B_3 \approx 0$ and $0 < X_2 < X_1 < 1$ not fulfilled:
  C_0 **regular**, standard BT not applicable
  Solution: Taylor expansion $\oplus$ special BT - procedure.

- $B_3 \approx 0$ and $0 < X_2 < X_1 < 1$ fulfilled:
  C_0 **singular**, standard BT not applicable
  Solution: Mellin - Barnes splitting $\oplus$ Sector decomposition $\oplus$ BT - algorithm

For more general cases there is a tree of B coefficients:

$$D_0(n = 4) = \frac{1}{2\, B_4}\{D_0(n = 6) \oplus \text{Combination of reduced}\, C_0\},$$

At the next iteration each reduced C_0 brings a **sub-leading** B_3 and the fulll chain $B_{\text{leading}} = B_3, B_{i,\text{subl}} = B_{3,i}$ must be treated.

NB cancellation of $E_0(n = 6)$ & $F_0(n = 6)$ trivial.

Around $B_G = 0$, **Mellin - Barnes technique**

$$I(\alpha) \; = \; \int_0^1 dx \, [\chi_B(x) - i\,\delta]^{-\alpha}.$$

$$\chi_B(x) \; = \; -P^2\,(x-a)^2 + \frac{\lambda}{4\,P^2} - i\,\delta, \quad a = \frac{1}{2}\,\frac{P^2 + m_2^2 - m_1^2}{P^2}.$$

A **pinch** will occur for $\lambda(-P^2, m_1^2, m_2^2) = 0$. Assume $P^2 = -s$, $s \geq 0$ and $0 \leq a \leq 1$ and obtain

$$I(\alpha) \; = \; (\frac{\rho}{s})^{\alpha} \sum_{X=a}^{1-a} \int_0^X dx \, (1 + \rho\, x^2)^{-\alpha}$$
$$= \; (\frac{\rho}{s})^{\alpha} \sum_{X=a}^{1-a} J_X(\alpha, \rho),$$

where $\rho = -4\,s^2/\lambda$. We are now interested in the behavior for $|\rho| \to \infty$. Consider the case $\lambda < 0$:

$$j_X(\alpha, s) \; = \; \int_0^\infty d\rho \, \rho^{s-1}\, J_X(\alpha, \rho) = \frac{B(s, \alpha - s)}{1 - 2\,s}\, X^{1-2\,s},$$

which is defined for $0 < \mathrm{Re}\, s < \min\{\frac{1}{2}, \mathrm{Re}\,\alpha\}$, $|\arg x^2| < \pi$.
The **inverse Mellin transform** will be

$$J_X(\alpha, \rho) \; = \; \frac{1}{2\,\pi\,i} \int_{-i\,\infty}^{+i\,\infty} ds \, \rho^{-s}\, j_X(\alpha, s).$$

The integration contour will be closed in the right s half-plane and **leading behavior** comes from the pole at $s = 1/2$,

$$J_X(\alpha, \rho) \; = \; -\frac{1}{2\,\pi\,i}\,\frac{B(1/2, \alpha - 1/2)}{2} \int_{-i\infty}^{+i\infty} ds \, [\frac{\rho^{-s}}{s - \frac{1}{2}} + \ldots]$$
$$= \; \frac{1}{2}\, B(\frac{1}{2}, \alpha - \frac{1}{2})\, \rho^{-\mathbf{1/2}} + \ldots, \quad 1 < \rho < \infty.$$

Any **realistic** numerical strategy must be able to deal with **IR** configurations, **Collinear** configurations, **Sudakov** configurations.

The **numerical** way to **IR** configurations: A necessary condition for the presence of infrared divergences is that **the Landau equations are fulfilled**.
We **regularize** in n-dimensions and isolate the **infrared poles** by using the technique of

iterated sector decomposition (or **defoliation**)

A k/n defoliation[3] is defined by the following relation

$$\int_0^1 \prod_{i=1}^{k} dx_i \prod_{j=k+1}^{n} dy_j\, F(\{x\}\,;\,\{y\}) = \sum_{l=1}^{k} \int_0^1 \prod_{i=1}^{k} dx_i\, x_l^{k-1}$$
$$\times \prod_{j=k+1}^{n} dy_j\, F(x_l\, x_1, \dots, x_l, \dots x_l\, x_k\,;\,\{y\}).$$

A chain of defoliations will stop when all the singularites of F which are located at the vertices of the **simplex** $[0,1]^k$ have been factorized,

$$F(x_l\, x_1, \dots, x_l, \dots x_l\, x_k\,;\,\{y\}) = x_l^{\mu_l}\, F(x_1, \dots, 1, \dots\, x_k\,;\,\{y\}).$$

To find the **sectors**, although far from unique, is **not** the difficult part of the evaluation. The resulting expression will contain **infrared poles** as well as an **infrared finite** part.
The latter and the residues are given by integrals of **inverse powers** of polynomials that are not (usually) **positive defined**. BT - iterations or Mellin - Barnes **splittings** are still required.

[3] T. Binoth and G. Heinrich, Nucl. Phys. B **585** (2000) 741 [arXiv:hep-ph/0004013]

For a complete understanding of how to implement **numerical integration** after sector decomposition we learn from

simple examples:

$$C_0\left(p_1^2, p_2^2, P^2; m, 0, m\right)$$

For $m_2 = 0$ and $m_1 = m_3 = m$ the **anomalous threshold** condition is satisfied if $p_1^2 = p_2^2 = -m^2$ and P^2 is uncontrained.

- Sector decomposition allows us to extract the **infrared pole** and the **IR finite part**, for simplicity here is the residue:

$$\begin{aligned} P_{IR} &= 2\,\mu^{\epsilon}\,\pi^{-\epsilon/2}\int_0^1 dx\int_0^1 dy\,x^{-1-\epsilon}\,\chi_{IR}^{-1-\epsilon/2} \\ &= -\frac{2}{\epsilon}\,\mu^{\epsilon}\,\pi^{-\epsilon/2}\int_0^1 dy\,\chi_{IR}^{-1-\epsilon/2}, \\ \chi_{IR}(y) &= m^2\,y^2 + (P^2 + 2\,m^2)\,y + m^2. \end{aligned}$$

Before attempting **numerical integration** one has to realize that χ_{IR} is **not positive definite** and, therefore, one has to seek for

$\boxed{\textbf{Strategies}\text{ for } P^2 = -\,s,\, s > 0:}$

- Away from $s \approx 4\,m^2$ and $m^2 \ll s$: one **BT** iteration is enough and **numerical integration** ($\to$ logarithms) follows;

- Around threshold: with $\beta^2 = 1 - 4\,m^2/s$ rewrite

$$\chi_{IR}(y) \;=\; m^2\,(y - \frac{1+\beta^2}{1-\beta^2})^2 + \frac{\beta^2}{\beta^2-1}\,s$$

and apply Mellin - Barnes techniques to obtain the $\beta \to 0$ behavior.

(N.B. infrared $\oplus$ threshold is the true, leading, Landau singularity)

- For $s \gg m^2$ (**collinear regime**) it is better to isolate the leading behavior, i.e. the **collinear** logs:

$$\int_0^1 dy\,\chi_{IR}^{-1} \quad \Rightarrow \quad \text{Mellin - Barnes}$$

$$= \frac{1}{4\,m^2}\,\frac{\Gamma\,(1/2)}{2\,\pi\,i}\int_{-i\,\infty}^{+i\,\infty} dt\,\frac{\Gamma\,(-t)\;\Gamma^2\,(t+1)}{\Gamma\,(t+3/2)}\,(-\frac{s}{4\,m^2})^t$$

$$m^2 \ll s \quad \Rightarrow \; t = n \;\; \text{poles}$$

$$= \; -\frac{1}{s}\sum_{n=0}^{\infty}\frac{\Gamma\,(1/2)}{\Gamma\,(1/2-n)\;n!}\,(-\frac{4\,m^2}{s})^n$$

$$\times \;\; [\ln\frac{-\,s}{4\,m^2} + 3\,\psi(n+1) - \psi(\frac{1}{2}-n)].$$

About **Reduction** and **Strategy**

The **standard** procedure is based on the introduction of **scalar functions** and of **tensor integrals** that are reduced to linear combinations of scalar ones.
The **new** procedure is different since the **functional relation** can be written in a more general form starting from

$$\int_S dx\, Q(x) \prod_i V_i^{\mu_i}(x), \quad Q(x) \;=\; \text{an arbitrary polynomial of}\, x.$$

Hence we are led to consider **all integrals** in **parametric space** with no **pre-reduction** in **momentum space**.

The final strategy will be to **map** as many integrals as possible in one integral,

$$\text{Process} \;\equiv\; \int_{\text{momenta}} \sum \text{diagrams} \;\equiv$$

$$\underbrace{\int_{S_{\max}} \sum_{ij} Q_{ij}(x_{\text{sub}_i} \;\in\; \{x_{\max}\})\, V_i^{\mu_i}(x_{\text{sub}_i} \;\in\; \{x_{\max}\})}_{\boxed{\text{one stroke}}} \;\otimes\; FF$$

Example: $Z \to \overline{f} f$, one-loop : $\{x_{\max}\} = \{x_1, x_2\}$,

$$Diagrams = \begin{cases} \mathbf{WFnormalization} \;\equiv\; \mathbf{two-point} & \mathbf{x_{sub_i} = x_1} \\ \mathbf{Vertices} \;\equiv\; \mathbf{three-point} & \mathbf{x_{sub_i} = x_{max}} \end{cases}$$

$$\text{Maximal simplex} \equiv S_{\max} \;=\; \{0 \leq x_1,\, x_2 \leq 1\}$$

$Q_{ij} =$ **jth component of i-diagram, without mom-reduction.**

Numerica

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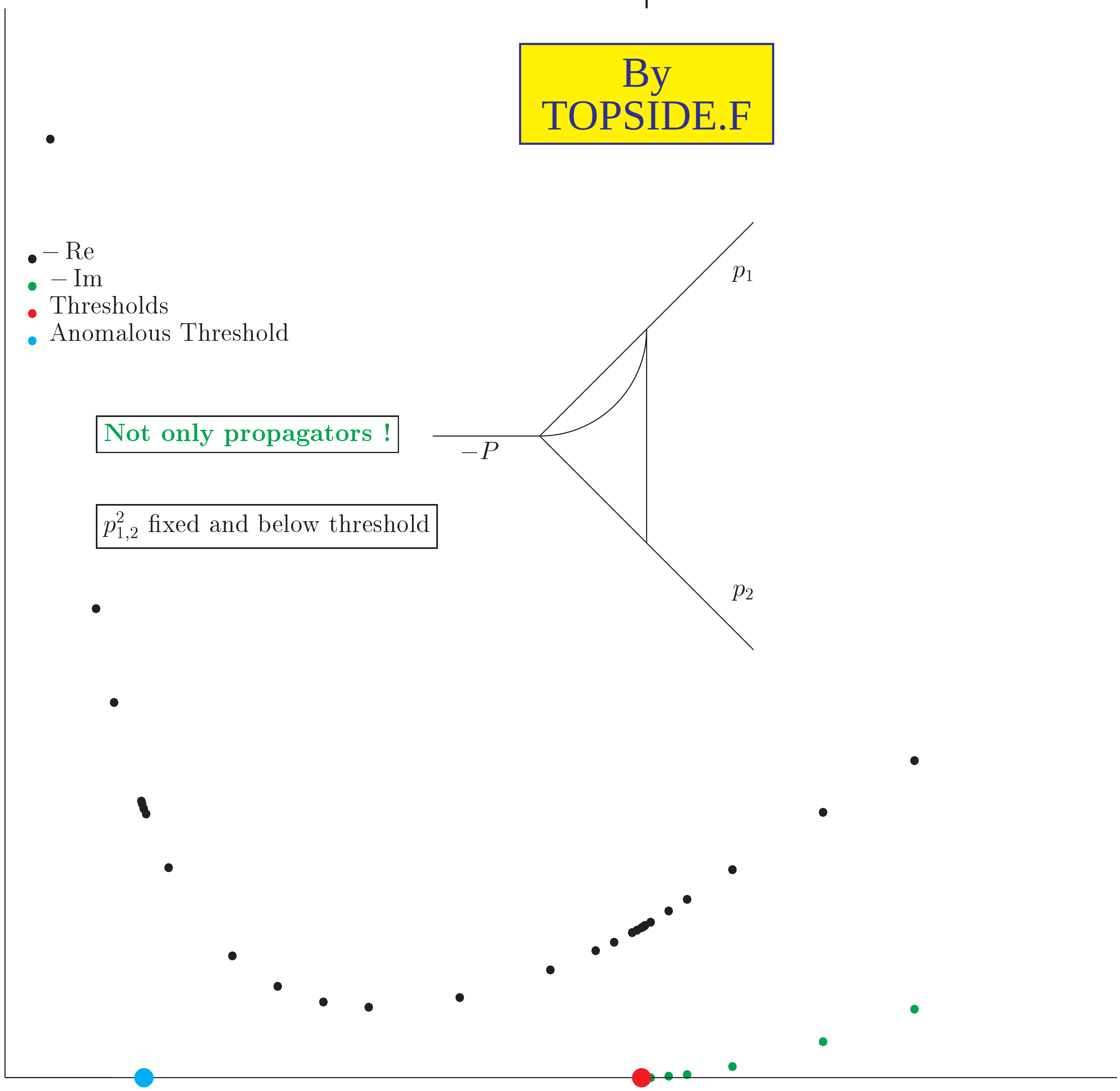
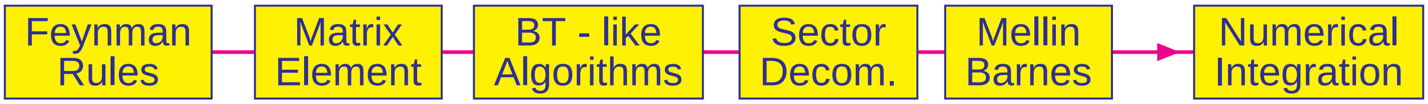
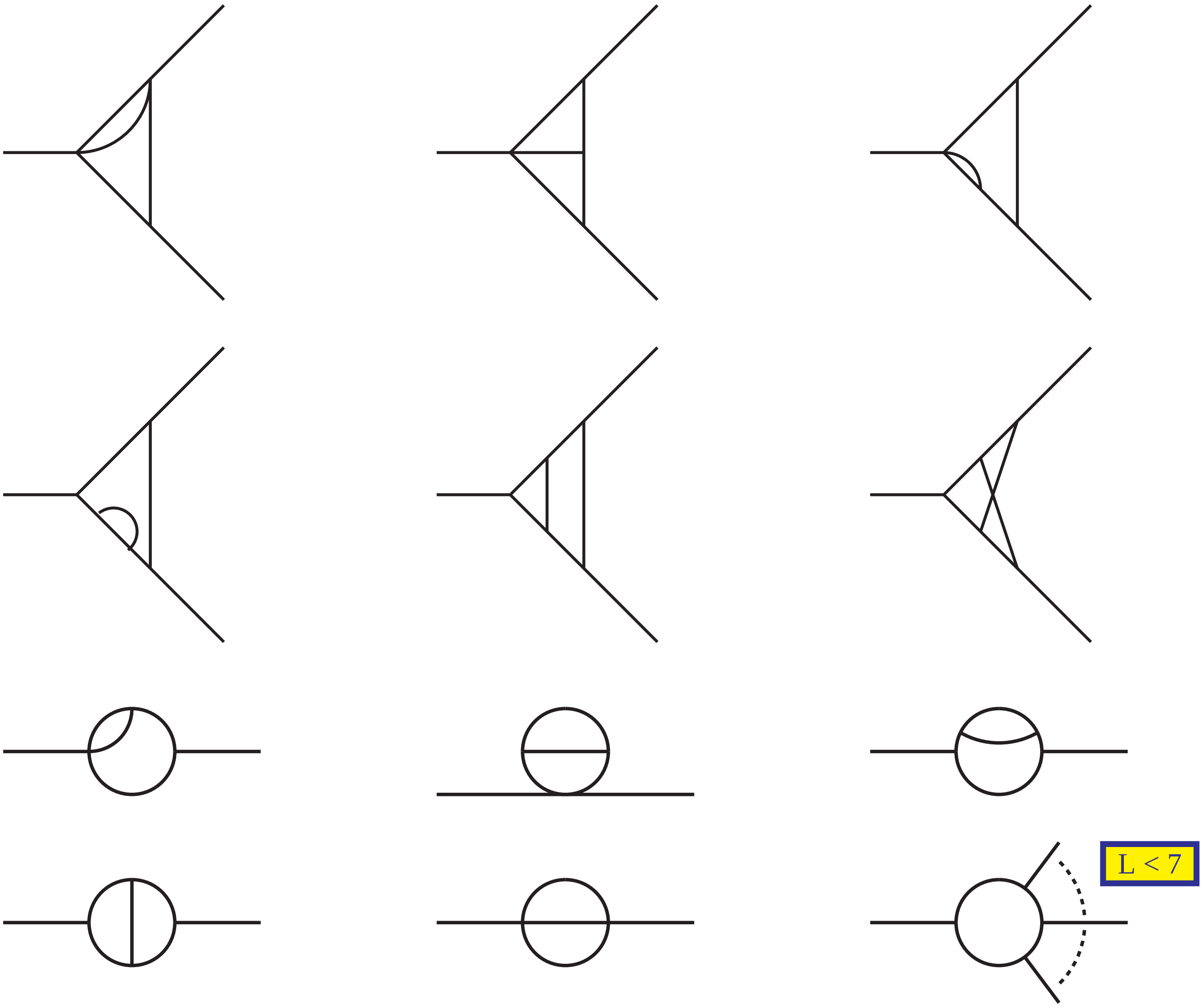
By TOPSIDE.F



Anomalous Threshold may occur below normal ones!!!





Availability

- ∇ Up to 9 different scales;
- ∇ Up to 6 normal thresholds (⊕ pseudo and anomalous);
- ∇ Unconstrained **kinematics**, **IR** - configurations accessible.