All-orders infra-red freezing of $R_{e^+e^-}$ in perturbative QCD

D. M. Howe* and C. J. Maxwell†

IPPP,
University of Durham

ICHEP 2002 Amsterdam, 25th July
hep-ph/0204036 v2, Phys. Lett. B in press.

*email:d.m.howe@durham.ac.uk
†email:c.j.maxwell@durham.ac.uk

What is $R_{e^+e^-}$?

$$R_{e^+e^-}(s) \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

$$R_{e^+e^-}(s) = 3\sum_f Q_f^2 (1 + \mathcal{R}(s))$$

Where $\mathcal{R}(s)$ can be calculated in perturbative QCD, and has been calculated to NNLO, so r_1 and r_2 are known

$$\mathcal{R}(s) = a + \sum_{n>0} r_n a^{n+1} .$$

Here $a \equiv \alpha_s(\mu^2)/\pi$.

What is infra-red freezing?

Infra-red freezing is having well-behaved, finite behaviour in the infra-red (low energy) limit.

$\mathcal{R}(s) \to \mathcal{R}^*$ as $s \to 0$

Can consider the s-dependence of $\mathcal{R}(s)$ at NNLO.

$$s\frac{d\mathcal{R}(s)}{ds} = -\frac{b}{2}\rho(\mathcal{R}) \equiv -\frac{b}{2}\mathcal{R}^2(1 + c\mathcal{R} + \rho_2\mathcal{R}^2)$$

Here $b = (33 - 2N_f)/6$, and $c = (153 - 19N_f)/12b$,are the first two universal QCD beta-function coefficients. The condition for $\mathcal{R}(s)$ to approach the infra-red limit $\mathcal{R}^*$ as $s \to 0$ is for the Effective Charge beta-function $\rho(\mathcal{R})$ to have a non-trivial zero, $\rho(\mathcal{R}^*) = 0$. At NNLO the condition for such a non-trivial zero is $\rho_2 < 0$. Putting $N_f = 2$ active flavours we find for the NNLO RS-invariant $\rho_2 = -9.72$. So that $\mathcal{R}(s)$ freezes in the infra-red to $\mathcal{R}^* = 0.43$. (Mattingly and Stevenson)

Should we believe this apparent NNLO freezing ? In fact ρ_2 is dominated by a large $b^2\pi^2$ term arising from Analytical Continuation (AC) of the Euclidean Adler $D(-s)$ function to the Minkowski $R(s)$, $\rho_2 = 9.40 - \pi^2 b^2/12$,similarly ρ_3 will contain the large AC term $-5c\pi^2 b^2/12$.

Thus to check freezing we should *resum* the AC terms to *all-orders*. The Adler D-function is the logarithmic energy derivative of the correlator of two vector currents, $\Pi(s)$,

$$D(s) = -s\frac{d}{ds}\Pi(s)\,.$$

The Minkowskian R(s) is obtained by analytical continuation of the perturbative corrections to $D(-s)$,

$$\mathcal{R}(s) = \frac{1}{2\pi i}\int_{-s-i\epsilon}^{-s+i\epsilon} dt\frac{\mathcal{D}(t)}{t} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\mathcal{D}(se^{i\theta})$$

Where $\mathcal{D}$ has the perturbation series,

$$\mathcal{D}(s) = a + \sum_{n>0} d_n a^{n+1}$$

Expanding $\mathcal{D}(se^{i\theta})$ in powers of $a(se^{i\theta})$ and integrating term-by-term we then obtain the "contour-improved" perturbation series,

$$\mathcal{R}(s) = A_1(s) + \sum_{n=1}^{\infty} d_n A_{n+1}(s)$$

The functions $A_n(s)$ resum at each order an infinite subset of AC terms present in the conventional perturbation series for $\mathcal{R}(s)$,

$$A_n(s) \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta a^n(se^{i\theta})$$

For the simplified case of a one-loop coupling one has,

$$a(s) = \frac{2}{b\ln(s/\tilde{\Lambda}^2_{\overline{MS}})}$$

The integrals are straightforward and one obtains,

$$\begin{aligned} A_1(s) &= \frac{2}{\pi b}\arctan\left(\frac{\pi b a(s)}{2}\right) \\ A_n(s) &= \frac{2a^{n-1}(s)}{b\pi(1-n)} Im\left[\left(1+\frac{ib\pi a(s)}{2}\right)^{1-n}\right] \quad (n>1) \end{aligned}$$

We then obtain the one-loop "contour-improved" series for $\mathcal{R}(s)$,

$$\begin{aligned}\mathcal{R}(s) &= \frac{2}{\pi b}\text{arctan}\left(\frac{\pi b a(s)}{2}\right) + d_1 \frac{a^2(s)}{(1+b^2\pi^2 a^2(s)/4)} \\ &+ d_2 \frac{a^3(s)}{(1+b^2\pi^2 a^2(s)/4)^2} + \ldots\end{aligned}$$

As $s \to \infty$ the $A_n(s)$ vanish as required by Asymptotic Freedom. However, although the one-loop coupling has a "Landau Pole" at $s = \tilde{\Lambda}^2_{\overline{MS}}$, the $A_n(s)$ are well-defined for all real s.

As $s \to 0$, $A_1(s)$ smoothly approaches from below the infra-red value $2/b$, whilst for $n > 1$ the $A_n(s)$ vanish. Thus in the infra-red limit $\mathcal{R}(s)$ is asymptotic to $\mathcal{R}(0) = 2/b$ to all-orders in perturbation theory.

C.f. Analytic Perturbation Theory (APT) approach of Shirkov, Solovtsov et. al.

We now turn to realistic QCD, beyond the one-loop approximation. It is convenient to work in an 't Hooft scheme, where the non-universal beta-function coefficients are all zero, and the beta-function equation has its two-loop form,

$$\frac{\partial a(\mu^2)}{\partial \ln \mu^2} = -\frac{b}{2} a^2(\mu^2)(1 + c a(\mu^2))$$

Crucially in such an RS we can express $a(\mu^2)$ analytically, in closed-form as

$$a(\mu^2) = -\frac{1}{c[1 + W_{-1}(A(\mu^2))]}$$

$$A(\mu^2) \equiv -\frac{1}{e}\left(\frac{\mu^2}{\tilde{\Lambda}^2_{\overline{MS}}}\right)^{-b/2c}$$

Here W denotes the Lambert W function, defined implicitly by $W(z)\exp(W(z)) \equiv z$. The "-1" subscript denotes the branch of the W function required for Asymptotic Freedom.

Magradze; Gardi, Grunberg and Karliner

Choosing a renormalization scale $\mu^2 = xs$ we can then evaluate the functions $A_n(s)$ in closed-form in terms of the W function

$$A_n(s) \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta a^n(xse^{i\theta})$$

Expressing the integrand in terms of the W function,

$$a(xse^{i\theta}) = -\frac{1}{c[1 + W(A(xse^{i\theta}))]}$$

, and making the change of variable $w = W(A(xse^{i\theta}))$, we then arrive at the elementary integral

$$A_{n+1}(s) = \frac{(-1)^n}{ibc^n\pi}\int_{l^*(s)}^{l(s)} \frac{dw}{w(1+w)^n}$$

$$\begin{aligned} l(s) &= W_{-1}(A(xse^{i\pi})) \\ l^*(s) &= W_1(A(xse^{-i\pi})) \end{aligned}$$

Thus we obtain the explicit results

$$A_1(s) = \frac{2}{b} + \frac{2}{\pi b} Im[\ln(l(s))]$$

$$A_{n+1}(s) = \frac{(-1)^n 2}{c^n b\pi} Im\left[\ln\left(\frac{l(s)}{1+l(s)}\right) + \sum_{k=1}^{n-1} \frac{1}{k(1+l(s))^k}\right]$$

$l(s) = W_{-1}(A(xse^{i\pi}))$.

CJM and Mirjalili; Magradze

Provided that $(b/c) > 0$ corresponding to $N_f <$ 9 quark flavours, one finds that the $A_n(s)$ are well-defined for all real s, with $A_1(s)$ approaching the infra-red limit $2/b$ from below, and for $n > 1$ $A_n(s)$ vanishing, so that $\mathcal{R}(s)$ is asymptotic to $\mathcal{R}(0) = 2/b$ to all-orders in perturbation theory.

Since the freezing result holds to all-orders in perturbation theory it is RS-independent. The use of an 't Hooft scheme serves to make the freezing manifest.

For a fixed-order zero of the beta-function one would expect the asymptotic infra-red behaviour $\mathcal{R}(s) - \mathcal{R}^{*} \sim s^{\gamma}$ where γ is a critical exponent. For the freezing induced by resummation of AC terms one finds instead the steeper behaviour

$$\mathcal{R}(s) - \frac{2}{b} \approx \frac{-1/c - 2/b}{W_0(-A(s))}$$

Again this involves the ubiquitous Lambert W function.

[FIGURE]

IR renormalons in the infra-red limit

By itself all-orders perturbation theory is undefined due to infra-red (IR) renormalons. The renormalon ambiguities cancel against the non-logarithmic UV divergences of the Operator Product Expansion (OPE). The vanishing of the $A_n(s)$ for $n > 1$ in the infra-red, means that the IR renormalon ambiguities disappear, and hence presumably the non-logarithmic UV divergences of the OPE also vanish in the infra-red. Thus for Minkowskian quantities it appears that perturbative and non-perturbative effects are *separately* well-defined in the infra-red limit.

Phenomenological applications of the freezing result

The key conclusion is that the contour-improved (APT) version of perturbation theory remains well-defined in the infra-red limit, and if suitably corrected to include quark masses and thresholds, can be used to supplement low energy data on $R_{e^+e^-}(s)$ in estimating hadronic corrections to QED $\alpha(M_Z)$, and to the anomalous magnetic moment of the muon. The explicit analytical expressions for the $A_n(s)$ make this straightforward to implement.

If one applies a smearing procedure, such as that of Poggio, Quinn and Weinberg, to the contour-improved perturbation theory and the data for $R_{e^+e^-}$, one averages out non-perturbative resonances and should find good agreement.

D.M. Howe and CJM work in progress