Construction of Supersymmetric Nonlinear Sigma Models on Noncompact Calabi-Yau Manifolds with Isometry

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Plan of My Talk

1. Introduction
2. $\mathcal{N} = 2$ SUSY σ models in 2D
3. Ricci-flat Kähler Manifolds
4. Our New Results
5. Summary

This talk is based on the work with

T. Kimura (Osaka) and **M. Nitta** (Purdue):

1. Calabi-Yau Manifolds of Cohomogeneity One as Complex Line Bundles (hep-th/0202064)

2. Ann. Phys. **296** (2002) 347-370 (hep-th/0110216)

3. Nucl. Phys. **B623** (2002) 133-149, (hep-th/0108084)

4. Phys. Lett. **B518** (2001) 301-305 (hep-th/0107100)

5. Phys. Lett. **B515** (2001) 421-425 (hep-th/0104184)

1. Introduction

Nonlinear σ models in two dimensions:

$$\mathcal{L}(x) = \frac{1}{2} g_{mn}(\phi) \partial_\mu \phi^m \partial^\mu \phi^n$$

- take values on curved manifolds
- are renormalizable

  $\delta g_{mn} \propto R_{mn}$ Ricci curvature
- describe strings in curved space
  - ◇ Conformal Invariance
  - ◇ $\mathcal{N} = 2$ Supersymmetry

$\Rightarrow$ Nonlinear σ models on Ricci-flat manifolds with $\mathcal{N} = 2$ supersymmetry

2. $\mathcal{N}=2$ SUSY Nonlinear σ Models in $2D$

$A^a(x)$: complex scalar, $\psi^a(x)$: complex fermion

$$\mathcal{L}(x) = g_{a\bar{b}}\partial_\mu A^a \partial^\mu \bar{A}^{\bar{b}} + i g_{a\bar{b}} \bar{\psi}^{\bar{b}} (\not{D}\psi)^a + \frac{1}{4} R_{a\bar{b}c\bar{d}} \psi^a \psi^c \bar{\psi}^{\bar{b}} \bar{\psi}^{\bar{d}}$$

$$g_{a\bar{b}}(A,\bar{A}) = \frac{\partial^2 K(A,\bar{A})}{\partial A^a \partial \bar{A}^{\bar{b}}} = K_{,a\bar{b}}(A,\bar{A})$$

$K(z,\bar{z})$: Kähler potential

manifold: Kähler manifold

$$\Gamma^c{}_{ab} = g^{c\bar{d}} g_{b\bar{d},a} = g^{c\bar{d}} K_{,ab\bar{d}}$$

$$(D_\mu \psi)^a = \partial_\mu \psi^a + \partial_\mu A^b \Gamma^a{}_{bc} \psi^c$$

3. Ricci Flat Kähler Manifolds

$$R_{a\bar{b}} = -\partial_a \partial_{\bar{b}} \log \det(g_{c\bar{d}}) = 0 \quad \text{PDE}$$

Use symmetry to reduce PDE to ODE!

Ricci-flat $\Longleftrightarrow$ $\boxed{\det(g_{c\bar{d}}) = const. \times |f(A)|^2}$

Eg: Ricci-flat mfd. with $U(N)$ symmetry

$$\vec{\phi} = \begin{pmatrix} \phi^1 \\ \cdots \\ \phi^{N-1} \\ \phi^N \end{pmatrix} = \sigma \begin{pmatrix} \varphi^1 \\ \cdots \\ \varphi^{(N-1)} \\ 1 \end{pmatrix} \quad \text{(Calabi)}$$

Assumption: Kähler Potential $K = K(X)$

is a function of $X = \log(\vec{\phi}^{\dagger}\phi) = \log|\sigma|^2 + \Psi$

where $\Psi = \log(1 + \Sigma_i |\varphi_i|^2)$

Fubini-Study metric of CP^{N-1}

$$M: \ \phi^i = \sigma\varphi^i \ (i = 1, \cdots, N-1), \ \phi^N = \sigma$$

$$\det(g_{a\bar{b}}) = |\sigma|^{2N-2} \underbrace{\mathrm{e}^{-NX} K''(K')^{N-1}}_{const.}$$

$$\Rightarrow \quad \boxed{K'(X) = (\lambda e^{NX} + b)^{\frac{1}{N}}}$$

Regular coordinate: (ρ, φ^i) where $\rho = \frac{(\sigma)^N}{N}$

When $\rho \to 0$, $M \sim \mathbf{C} \times CP^{N-1}$ of radius $b^{1/N}$

When $\rho \to \infty$, $M \sim \mathbf{C}^N / Z_N$

Our New Result

We have constructed noncompact Ricci-flat mfd. M for each hermitian symmetric space (HSS) G/H, by introducing one additional complex coordinate ρ.

Kähler potential of M: $$K'(X) = (\lambda e^{X} + b)^{\frac{1}{D}}$$

$X = \log|\rho|^2 + h\Psi(\varphi, \bar{\varphi})$, G-invariant (Nibbelink,2000)

D: complex dimensions of M

$h = \frac{C_2(G)}{2}$ (dual Coxeter number of G)

$\Psi(\varphi, \bar{\varphi})$: Kähler potential of HSS (IKK, 1986)

Hermitian Symmetric Spaces (HSS)

= Kähler G/H which satisfy

$[\text{broken}, \text{broken}] \subset \text{unbroken}$

Hermitian Symmetric Spaces

G/H	$D = dim(G/H) + 1$	h
$SU(N)/SU(N-1) \times U(1)$	N	N
$U(N)/U(N-M) \times U(M)$	$M(N-M)$	N
$SO(N)/SO(N-2) \times U(1)$	$N-1$	$N-2$
$Sp(N)/U(N)$	$\frac{1}{2}N(N+1)+1$	$N+1$
$SO(2N)/U(N)$	$\frac{1}{2}N(N-1)+1$	$N-1$
$E_6/SO(10) \times U(1)$	17	12
$E_7/E_6 \times U(1)$	28	18

$$\boldsymbol{C}P^{N-1} = SU(N)/SU(N-1) \times U(1)$$

$$G_{N,M}(\boldsymbol{C}) = U(N)/U(N-M) \times U(M)$$

$$Q^{N-2}(\boldsymbol{C}) = SO(N)/SO(N-2) \times U(1)$$

Summary and Discussion

- Noncompact Ricci-flat mfd. with isometry G are constructed by adding a complex coordinate to Hermitian Symmetric Spaces
  - metrics are known explicitly!
- Can be generalized to G/H Kähler-Einstein manifold ($R_{i\bar{j}} = hg_{i\bar{j}}$)
  - Page-Pope (1987), HKN (2002)
- Further Fun?
  - What is the corresponding CFT?
  - Relation to Kazama-Suzuki Models?
  - Off-critical Behavior?

Examples

- Isomorphism
  - $CP^1 \simeq O(4)/U(2) \simeq Sp(1)/U(1) \simeq Q^1$
  - $Q^2 \simeq CP^1 \times CP^1$
  - $CP^3 \simeq SO(6)/U(3)$
  - $Sp(2)/U(2) \simeq Q^3$
  - $G_{4,2} \simeq Q^4$

- Lower dimensional manifolds
  - $C \times CP^1 \;\Rightarrow\;$ Eguchi-Hanson (1978)
  - $C \times Q^2 \;\Rightarrow\;$ Conifold

    Candelas-de la Ossa (1990)

    Pando Zayas-Tseytlin (2000)

$\mathcal{N} = 2$ SUSY in 2-dimensions

• **Superspace**

$$\begin{array}{lll} x^\mu & \mu = 0, 1 & \text{Bosonic coordinates} \\ \theta^\alpha & \alpha = 1, 2 & \text{complex spinor} \end{array}$$

• **Chiral Superfield**

$$\phi(x,\theta) = A(x) + \theta\psi(x) + \frac{1}{2}\theta^2 F(x)$$

$$\begin{array}{ll} A(x) & \text{complex scalar} \\ \psi(x) & \text{complex fermion} \\ F(x) & \text{auxiliary field} \end{array}$$

- **Real Superfield (Gauge Superfield)**

$$
\begin{aligned}
V(x,\theta,\bar\theta) \;&= \phi(x) + (\theta\psi + \bar\theta\bar\psi) \\
&+\tfrac{1}{2}\bar\theta\gamma_\mu\theta V^\mu(x) + \tfrac{1}{2}\bar\theta\theta S + \tfrac{1}{2}\bar\theta i\gamma_3\theta P \\
&+\tfrac{1}{4}(\bar\theta^2\theta\lambda + \theta^2\bar\theta\bar\lambda) + \tfrac{1}{16}\theta^2\bar\theta^2 D(x)
\end{aligned}
$$

$V^\mu(x)$	Gauge Field;	S	Scalar
$\lambda(x)$	Gaugino	P	Pseudo Scalar

- **SUSY Lagrangian:**

  D-term $\{K(\phi^*,\phi)\}_D$

  F-term $\{W(\phi)\}_F$

$$\mathcal{L}(x) = \{K(\phi^*,\phi)\}_D + \{W(\phi)\}_F + h.c.$$

Riemann Curvature

$$R^{\bar{a}}{}_{\bar{b}c\bar{d}} = \partial_c \Gamma^{\bar{a}}{}_{\bar{b}\bar{d}} = \partial_c (g^{m\bar{a}} g_{m\bar{b},\bar{d}})$$

$$\begin{aligned} R_{a\bar{b}c\bar{d}} &\equiv g_{a\bar{m}} R^{\bar{m}}{}_{\bar{b}c\bar{d}} \\ &= K,_{a\bar{b}c\bar{d}} - g^{m\bar{n}} K,_{m\bar{b}\bar{d}} K,_{\bar{n}ac} \end{aligned}$$

Rcci Curvature

$$R_{a\bar{b}} = -g^{c\bar{d}} R_{c\bar{d}a\bar{b}} = -\partial_a \partial_{\bar{b}} \log \det(g_{c\bar{d}})$$